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A STUDY OF THE STARS OF SPECTRAL TYPE A

BY

K. GUNNAR MALMQUIST

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AMSTERDAM

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INTRODUCTION.

1. The greatest difficulty met with in the researches into the structure of our stellar universe is the imperfection of our knowledge of the distances of the stars. The direct determinations of stellar parallaxes are few in number and their execution very laborious and time-wasting, so that we cannot hope in that way to ascertain the distances of any large number of stars in a reasonably near future. As regards the later spectral types, however, matters have been considerably improved in the last few years thanks to the interesting spectroscopical method proposed by ADAMS and KOHL-SCHÜTTER and further developed by ADAMS. Unfortunately this method—in its present state at least—is not applicable to the spectral types B and A. For these types we are reduced to more indirect determinations. The most commonly used method in this case is to determine the mean parallax. The mean parallax, however, is a very defective measure even of the mean distances of the stars, as the small distances play a dominative part at the expence of the great ones.

As pointed out by CHARLIER it is therefore more advantageous to introduce instead of the distance r of a star another parameter R , which will be defined in the first chapter. This new parameter, which is a function of the absolute magnitude only, offers a great advantage also from a theoretical point of view. From the proper motions and apparent magnitudes of the stars the mean value of $\frac{1}{R}$ may be calculated in quite the same way as the mean parallax from the proper motions.

2. This method I have applied to the stars of the spectral types B8 to A5. In the first chapter, which is chiefly an account of the method and results published in L. M. I. 76, I have, just as CHARLIER concerning the B-stars in L. M. II, 14, assumed that the parameter R is a constant for each subclass. The method of computing R is given in its details, as the same method—with some modification—is also used in the third chapter.

In the second chapter I have given up the assumption that R —and then also the absolute magnitude—is a constant, and assume that the frequency function of the absolute magnitudes of the stars in question is a normal curve of type A. (The denomination emanates from CHARLIER, L. M. II, 4.) Starting from this it is shown that *without making any special assumption concerning the density function* the mean value of any power of R is a very simple function of the observed distribution of the apparent magnitudes and the mean value, M_0 , and the dispersion, σ , of the absolute magnitudes. From the known mean value of $\frac{1}{R}$ we thus get a simple relation between M_0 and σ . The method gives us a possibility—under certain conditions—to determine the values of these two constants.

The results of the theoretical treatment are put in practice in the third chapter, where the relations between the two constants are derived and an attempt at their determination is made, although the material is too scanty to give any reliable results. We can only say that the dispersion in the absolute magnitude must be small, a result supported by the mean parallaxes of stars down to the ninth magnitude, from which we get a value of about $0^m.9$. For this value of σ the corresponding mean magnitude is found to be $-0^m.2$.

In the fourth chapter the velocity distribution is studied with the help of the ellipsoidal hypothesis. On account of the relatively small material I have confined myself to the moments of the first and second orders.

3. The data used in this investigation are:—

- (a) The coordinates and proper motions taken for stars brighter than the sixth magnitude from BOSS's *Preliminary General Catalogue*, for the fainter stars from *The Greenwich Catalogue of Stars for 1910.0, Part. II* (not yet published).
- (b) The apparent magnitudes and spectra for the BOSS-stars taken from *The Revised Harvard Photometry* (H. A. 50), for the stars in the Greenwich Catalogue the magnitudes and spectra are given there.
- (c) The coordinates, apparent magnitudes and spectra taken from *The HENRY DRAPER Catalogue*, 0^h , 1^h , 2^h and 3^h (H. A. 91).
- (d) The radial velocities used in the first chapter taken from the catalogue of Dr. GYLLENBERG (L. M. II, 13).

4. The units of length and time are Siriometer and Stellar Year defined by CHARLIER as follows

- 1 Siriometer (Sir.) = 10^6 times the mean distance of the sun from the earth.
- 1 Stellar Year (St.) = 10^6 tropical years.

5. The following abbreviations are used

- A. J. = The Astronomical Journal.
- Ap. J. = The Astrophysical Journal.
- A. N. = Astronomische Nachrichten.
- G. P. = Publications of the Astronomical Laboratory at Groningen.
- H. A. = Annals of the Astronomical Observatory of Harvard College.
- L. M. I = Meddelanden från Lunds Astronomiska Observatorium, Serie I.
- L. M. II = Meddelanden från Lunds Astronomiska Observatorium, Serie II.
- M. N. = Monthly Notices of the Royal Astronomical Society.

CHAPTER I.

RESULTS DERIVED UNDER THE ASSUMPTION OF A CONSTANT ABSOLUTE MAGNITUDE FOR EACH SUBCLASS.

6. Let M denote the absolute magnitude of a star (= apparent magnitude at the distance of one siriometer), m its apparent magnitude and r its distance, and we have the relation

$$r = 10^{0.2(m-M)}$$

under the supposition that there exists no appreciable absorption of light in space.

Putting

$$(1) \quad R = 10^{-0.2M},$$

we get

$$(1^*) \quad r = R \cdot 10^{0.2m}.$$

For $m = 0$ we get $r = R$. R is also the distance at which the star in question would have an apparent magnitude equal to zero.

7. As pointed out by CHARLIER¹⁾, it is more suitable to introduce the parameter R instead of r , because the variation in R —even if all spectral types are considered together—is much smaller than the corresponding variation in r . If we confine ourselves to the earlier spectral types, the advantage increases in a considerable degree. From the view of RUSSEL's²⁾ theory of giant and dwarf stars, the types B and A consist mainly of giant stars, for which the variation in absolute magnitude is supposed to be small. Also in a theoretical manner it can be shown—as done by EDDINGTON³⁾—that, under certain presumable conditions, the variation in absolute magnitude cannot be great for the giant stars. From a quite different point of view CHARLIER⁴⁾ has achieved the same result. He has shown that, if we assume the density to be constant—a permissive approximation when only the brighter stars are concerned—and the frequency function of the absolute magnitude to be a normal curve of type A, we may deduce from the mean parallaxes of the different spectral types a linear relation between M_0 and σ^2 , the constants in the frequency function of the absolute magnitude. From this relation it is seen that the dispersion must be smallest for the B-stars and then increase as we pass through the spectral scale.

¹⁾ Die Abstände der Sterne vom Spectral-Typus B. A. N. 201, 9, 1915.

²⁾ Relations between the Spectra and other Characteristics of the Stars. Nature, Vol. 93, 1914.

³⁾ On the Radiative Equilibrium of the Stars. M. N. 77, 16, 1916.

⁴⁾ On the Mean Distances and Luminosities of Stars of different Spectral Types. L. M. I, 68, 1915.

In consequence of these results, CHARLIER¹⁾ has treated the B-stars under the assumption that the absolute magnitude—and then also R —is a constant for each subclass, and in this way very important results are obtained. I have treated the subclasses B8—A5 under the same assumptions and the results are published in L. M. I, 76. Similar investigations have been made by GYLLENBERG²⁾ for the spectral type O, by LUNDAHL³⁾ for type F, by FÄNGE⁴⁾ for types G and M, and by KRANTZ and LUNDAHL⁴⁾ for the K-stars.

8. As the present investigation is an extension of the one published in L. M. I, 76, I will, in this chapter, give an account of the method employed, which will also be used in the following chapters, and of the results obtained there.

Put

$$\begin{aligned} u &= \Delta\alpha \cos \delta, \\ v &= \Delta\delta, \end{aligned}$$

where $\Delta\alpha$ and $\Delta\delta$ are the proper motions in right ascension and declination for a star, let further U and V denote the corresponding linear velocity components, and we have

$$\begin{aligned} U &= ru, \\ V &= rv. \end{aligned}$$

U and V are also the linear velocity components along the axes of X and Y in a system of coordinates having its axis of Z directed along the radius vector to the star, the axis of X in the direction of increasing α , the axis of Y in the direction of increasing δ . This system of coordinates is called K . The direction cosines relating the axes of this system (K) with those of the common astronomical equator system (K'') are given through the following scheme

	K'' :		
	X''	Y''	Z''
X	γ_{11}	γ_{21}	γ_{31}
$K: Y$	γ_{12}	γ_{22}	γ_{32}
Z	γ_{13}	γ_{23}	γ_{33}

where

$$\begin{aligned} \gamma_{11} &= -\sin \alpha; & \gamma_{21} &= +\cos \alpha; & \gamma_{31} &= 0; \\ \gamma_{12} &= -\sin \delta \cos \alpha; & \gamma_{22} &= -\sin \delta \sin \alpha; & \gamma_{32} &= +\cos \delta; \\ \gamma_{13} &= +\cos \delta \cos \alpha; & \gamma_{23} &= +\cos \delta \sin \alpha; & \gamma_{33} &= +\sin \delta; \end{aligned}$$

α and δ are the right ascension and declination of a star.

¹⁾ *Studies in Stellar Statistics III*. The Distances and the Distribution of the Stars of the Spectral Type B. L. M. II, 14, 1916.

²⁾ The Distribution in Space of Stars of the Spectral Type O. L. M. I, 75, 1917.

³⁾ Preliminary Notice on the Distances and Velocities of the Stars of Spectral Class F. L. M. I, 77, 1917. On some Properties of the Stars of Spectral Type F. L. M. II, 21, 1919.

⁴⁾ Not yet published.

With the help of the relation (1*) we get

$$U = R \cdot 10^{0.2 m} u,$$

$$V = R \cdot 10^{0.2 m} v.$$

Putting

$$u' = 10^{0.2 m} u,$$

$$v' = 10^{0.2 m} v,$$

the above relations take the following forms

$$(2) \quad \begin{aligned} U &= R u', \\ V &= R v'. \end{aligned}$$

The quantities u' and v' are called by CHARLIER the *reduced proper motions* of the star.

Let further U'' , V'' , W'' be the velocity components of the star in the common equatorial system and we have the relations

$$(3) \quad \begin{aligned} U &= \gamma_{11} U'' + \gamma_{21} V'' + \gamma_{31} W'', \\ V &= \gamma_{12} U'' + \gamma_{22} V'' + \gamma_{32} W'', \end{aligned}$$

where γ_{ij} are the direction cosines defined above.

Making use of (2) and observing that $\gamma_{31} = 0$ we get from (3)

$$(4) \quad \begin{aligned} u' &= \gamma_{11} \frac{1}{R} U'' + \gamma_{21} \frac{1}{R} V'', \\ v' &= \gamma_{12} \frac{1}{R} U'' + \gamma_{22} \frac{1}{R} V'' + \gamma_{32} \frac{1}{R} W''. \end{aligned}$$

Here u' , v' and the direction cosines are known and we form the following normal equations for each subclass

$$(4^*) \quad \begin{aligned} [\gamma_{11} u'] &= [\gamma_{11} \gamma_{11}] \frac{1}{R} U'' + [\gamma_{11} \gamma_{21}] \frac{1}{R} V'', \\ [\gamma_{21} u'] &= [\gamma_{21} \gamma_{11}] \frac{1}{R} U'' + [\gamma_{21} \gamma_{21}] \frac{1}{R} V'', \\ [\gamma_{12} v'] &= [\gamma_{12} \gamma_{12}] \frac{1}{R} U'' + [\gamma_{12} \gamma_{22}] \frac{1}{R} V'' + [\gamma_{12} \gamma_{32}] \frac{1}{R} W'', \\ [\gamma_{22} v'] &= [\gamma_{22} \gamma_{12}] \frac{1}{R} U'' + [\gamma_{22} \gamma_{22}] \frac{1}{R} V'' + [\gamma_{22} \gamma_{32}] \frac{1}{R} W'', \\ [\gamma_{32} v'] &= [\gamma_{32} \gamma_{12}] \frac{1}{R} U'' + [\gamma_{32} \gamma_{22}] \frac{1}{R} V'' + [\gamma_{32} \gamma_{32}] \frac{1}{R} W''. \end{aligned}$$

The corresponding normal equations obtained from the proper motions in right ascension and declination are then added together and we have to solve the following system of equations

$$\begin{aligned}
 [\gamma_{11} u'] + [\gamma_{12} v'] &= ([\gamma_{11} \gamma_{11}] + [\gamma_{12} \gamma_{12}]) \frac{1}{R} U'' + ([\gamma_{11} \gamma_{21}] + [\gamma_{12} \gamma_{22}]) \frac{1}{R} V'' + [\gamma_{12} \gamma_{32}] \frac{1}{R} W'' \\
 (5) \quad [\gamma_{21} u'] + [\gamma_{22} v'] &= ([\gamma_{21} \gamma_{11}] + [\gamma_{22} \gamma_{12}]) \frac{1}{R} U'' + ([\gamma_{21} \gamma_{21}] + [\gamma_{22} \gamma_{22}]) \frac{1}{R} V'' + [\gamma_{22} \gamma_{32}] \frac{1}{R} W'' \\
 [\gamma_{32} v'] &= [\gamma_{32} \gamma_{12}] \frac{1}{R} U'' + [\gamma_{32} \gamma_{22}] \frac{1}{R} V'' + [\gamma_{32} \gamma_{32}] \frac{1}{R} W''
 \end{aligned}$$

As we wish to express the unknown quantities in radians per stellar year and u' and v' are given in seconds of arc per year we have, before solving, to multiply the left members in the normal equations by

$$\frac{10^6}{206\,265} = 4.8481.$$

Assuming R to be a constant for each subclass we get from the equation (5) the values of

$$\frac{1}{R} \overline{U''}, \quad \frac{1}{R} \overline{V''}, \quad \frac{1}{R} \overline{W''},$$

where $\overline{U''}$, $\overline{V''}$, $\overline{W''}$ are the mean values of U'' , V'' , W'' . But if S is the velocity of the sun, we have

$$S^2 = \overline{U''^2} + \overline{V''^2} + \overline{W''^2}.$$

Hence it follows that

$$(6) \quad \frac{S}{R} = \sqrt{\left(\frac{1}{R} \overline{U''}\right)^2 + \left(\frac{1}{R} \overline{V''}\right)^2 + \left(\frac{1}{R} \overline{W''}\right)^2}.$$

With the help of the known velocity of the sun we then get the value of R .

9. The equations (5) are solved in the following way. If we write the normal equations in the form

$$\begin{aligned}
 [a n] &= [a a]x + [a b]y + [a c]z, \\
 [b n] &= [b a]x + [b b]y + [b c]z, \\
 [c n] &= [c a]x + [c b]y + [c c]z,
 \end{aligned}$$

we get the following expression for the unknown quantities

$$\frac{x}{D_{na}} = \frac{y}{D_{nb}} = \frac{z}{D_{nc}} = -\frac{1}{D_{nn}},$$

where D_{ij} are the subdeterminants of the determinant

$$D = \begin{vmatrix} [nn], [na], [nb], [nc] \\ [an], [aa], [ab], [ac] \\ [bn], [ba], [bb], [bc] \\ [cn], [ca], [cb], [cc] \end{vmatrix}$$

The mean errors in x, y, z are given from

$$\frac{\epsilon^2(x)}{D_{nn, aa}} = \frac{\epsilon^2(y)}{D_{nn, bb}} = \frac{\epsilon^2(z)}{D_{nn, cc}} = \frac{D}{ND_{nn}^2},$$

where N is the number of equations.

When the mean errors in

$$x \left(= \frac{1}{R} \bar{U}'' \right), y \left(= \frac{1}{R} \bar{V}'' \right)$$

and

$$z \left(= \frac{1}{R} \bar{W}'' \right)$$

are calculated in this way the mean error in $\frac{S}{R}$ is given from

$$\epsilon^2 \left(\frac{S}{R} \right) = \frac{x^2 \epsilon^2(x) + y^2 \epsilon^2(y) + z^2 \epsilon^2(z)}{\left(\frac{S}{R} \right)^2}$$

and from

$$R = \frac{S}{\frac{S}{R}},$$

we get

$$\epsilon^2(R) = \epsilon^2(S) \frac{R^2}{S^2} + \epsilon^2 \left(\frac{S}{R} \right) \frac{R^4}{S^2}.$$

10. The material of the investigation in L. M. I, 76 consists of all the stars brighter than the sixth magnitude and belonging to the subclasses B8—A5 according to H. A. 50 which are to be found in BOSS's *Preliminary General Catalogue*, from which the values of the proper motions are taken. The magnitudes are those of H. A. 50. One star of the subclass A5 was excluded on account of large reduced proper motion. The remaining number of stars is 1459.

First the direction cosines and the reduced proper motions were calculated for each star and then the normal equations (5) were formed and solved. The material was combined in three different manners, taking first all stars brighter than the fifth magnitude, then all between 5^m_{00} — 5^m_{99} and, finally, all stars brighter than the sixth magnitude.

As value of the velocity of the sun I use

$$\begin{aligned} S &= 4.167 \pm 0.305 \text{ Sir./St.} \\ &= 19.8 \pm 1.5 \text{ Km./Sec.} \end{aligned}$$

found by GYLLENBERG¹⁾ from the radial velocities of 291 stars of the spectral type A. This value is in good agreement with that obtained from all the types together.

The results are given in the following table.

¹⁾ Stellar Velocity Distribution as derived from Observations in the Line of Sight. L. M. II, 13, 1915.

TABLE I.

RESULTS OBTAINED UNDER THE ASSUMPTION OF A CONSTANT ABSOLUTE MAGNITUDE.

m	$\bar{U}''/R$	$\bar{V}''/R$	$\bar{W}''/R$	S/R	R	$\varepsilon(R)$	M	A	D	N
<i>Subclass B 8.</i>										
≤ 4.99	+ 0.0754	+ 1.7282	- 0.7510	1.89 ± 0.22	2.21	0.80	- 1.72	$267^{\circ}50$	+ $23^{\circ}42$	51
5.00—5.99	- 0.1709	+ 1.5460	- 1.1115	1.91 ± 0.18	2.19	0.26	- 1.70			87
≤ 5.99	- 0.1497	+ 1.6162	- 0.9871	1.90 ± 0.14	2.19	0.23	- 1.70	$275^{\circ}28$	+ $31^{\circ}30$	138
<i>Subclass B 9.</i>										
≤ 4.99	+ 0.1156	+ 1.5460	- 0.7013	1.70 ± 0.49	2.45	0.72	- 1.95	$265^{\circ}72$	+ $24^{\circ}36$	20
5.00—5.99	+ 0.2885	+ 1.8129	- 0.9033	2.05 ± 0.26	2.03	0.30	- 1.54			50
≤ 5.99	+ 0.2418	+ 1.7330	- 0.8701	1.95 ± 0.24	2.14	0.30	- 1.65	$262^{\circ}06$	+ $26^{\circ}50$	70
<i>Subclass A 0.</i>										
≤ 4.99	+ 0.0489	+ 1.6726	- 1.0794	1.99 ± 0.23	2.10	0.29	- 1.61	$268^{\circ}32$	+ $32^{\circ}85$	151
5.00—5.99	+ 0.0065	+ 2.3813	- 1.1547	2.65 ± 0.19	1.57	0.16	- 0.98			705
≤ 5.99	- 0.0037	+ 2.2542	- 1.1203	2.52 ± 0.16	1.66	0.16	- 1.10	$270^{\circ}10$	+ $26^{\circ}40$	856
<i>Subclass A 2.</i>										
≤ 4.99	+ 0.0577	+ 1.4518	- 0.9465	1.73 ± 0.40	2.41	0.58	- 1.87	$267^{\circ}72$	+ $33^{\circ}17$	79
5.00—5.99	- 0.0839	+ 2.0006	- 1.1237	2.80 ± 0.35	1.81	0.33	- 1.29			113
≤ 5.99	- 0.0161	+ 1.7754	- 1.0390	2.06 ± 0.26	2.02	0.30	- 1.53	$270^{\circ}77$	+ $31^{\circ}47$	192
<i>Subclass A 3.</i>										
≤ 4.99	+ 0.2757	+ 2.3411	- 0.8254	2.50 ± 0.87	1.67	0.54	- 1.11	$263^{\circ}28$	+ $19^{\circ}28$	23
5.00—5.99	+ 0.1071	+ 3.6420	- 1.1867	3.88 ± 0.78	1.09	0.26	- 0.19			39
≤ 4.99	+ 0.1615	+ 3.2311	- 1.0918	3.41 ± 0.63	1.22	0.24	- 0.43	$267^{\circ}14$	+ $18^{\circ}68$	62
<i>Subclass A 5.</i>										
≤ 4.99	+ 0.3888	+ 2.5740	- 0.8034	2.72 ± 0.76	1.53	0.44	- 0.92	$261^{\circ}41$	+ $17^{\circ}18$	52
5.00—5.99	+ 0.3275	+ 2.6340	- 1.1791	2.90 ± 0.63	1.44	0.33	- 0.79			89
≤ 5.99	+ 0.3475	+ 2.6101	- 1.0467	2.88 ± 0.49	1.47	0.26	- 0.84	$262^{\circ}42$	+ $21^{\circ}71$	141

The first seven columns in this table need no further explanation, the eighth column gives the absolute magnitude calculated from the relation

$$M = -5 \cdot {}^{10}\log R.$$

The next two columns give the right ascension and declination of the apex, obtained from the relations

$$(6^*) \quad \begin{aligned} \frac{\bar{U}''}{R} &= -\frac{S}{R} \cos D \cos A; \\ \frac{\bar{V}''}{R} &= -\frac{S}{R} \cos D \sin A; \\ \frac{\bar{W}''}{R} &= -\frac{S}{R} \sin D. \end{aligned}$$

For all the stars brighter than the fifth magnitude we get

$$\begin{aligned} A &= 266^{\circ}57, \\ D &= +28^{\circ}01 \end{aligned}$$

and for all the stars brighter than the sixth magnitude

$$A = 269^{\circ}_{36},$$

$$D = + 26^{\circ}_{71}.$$

11. A fact worth noticing is that for each subclass the value of R is altogether smaller for the faint stars than for the bright ones. That the value is not the same is, of course, due to the fact that our assumption of R constant is only an approximation. In fact we have, through the method given here, determined the mean value of $\frac{1}{R}$ for the stars in question. If we denote this mean value

$$\left(\frac{1}{R}\right),$$

our value of R given in table I is obtained from

$$R = \frac{1}{\left(\frac{1}{R}\right)}$$

which value need not be identical with the mean value of R , no more than the value of M in the table I, which is obtained from

$$M = 5 \cdot 10 \log \left(\frac{1}{R}\right)$$

need be identical with the mean value of the absolute magnitude for the stars in question. We have a similar case as regards the mean parallax. From the parallactic motion the mean value of $\frac{1}{r}$ is computed. In several astronomical papers the assumption is made that the mean value of the distance is equal to the invers value of the mean value of $\frac{1}{r}$ and also that the mean value of $\log r$ may be put equal to minus the logarithm of the same mean value. As pointed out by CHARLIER, these assumptions are, in a considerable degree, erroneous. The relations between the values of these different mean values are dependent on the density distribution and the distribution of the absolute magnitudes of the stars in question. In the next chapter I will treat this matter concerning the reduced motion.

12. Another fact from table I is worth noticing. If from the numbers in the last column we calculate the ratio of the number of stars brighter than the sixth magnitude to those brighter than the fifth magnitude we get the values given in table II.

TABLE II.

$A(5.99) : A(4.99)$ FOR THE DIFFERENT SUBCLASSES.

	B 8	B 9	A 0	A 2	A 3	A 5	B 8—A 5
$\frac{A(5.99)}{A(4.99)} \dots \dots$	2.71	3.50	5.67	2.43	2.70	2.71	3.88

The value of this ratio is considerably greater for the subclass A0 than for the others. The most probable reason for this fact is that for the fainter stars the increasing difficulty of a perfect separation of the different subclasses leads to the consequence that more stars are classified as A0. With regard to this fact and to the amount of the mean errors in R for the different subclasses, I have in the following chapters put together all the subclasses in one group.

13. Assuming R to be a constant, we get the distance of each separate star expressed in sirimeters from the relation (1*)

$$r = R \cdot 10^{0.2m}.$$

We may now, still assuming the value of R as constant, calculate the density around the sun of the stars belonging to the subclasses B8—A5. Using the values of R found from the stars brighter than the sixth magnitude, we see from table I that the subclass A3 gives us the lowest value of this parameter. All stars of this subclass brighter than the sixth magnitude are situated around the sun within a radius of about 20 sirimeters. Consequently, if we confine ourselves to stars brighter than the sixth magnitude, this is the greatest distance where all subclasses are completely represented. Within this distance we get the density

$$0.0218 \text{ stars per cub. sir.}$$

To give an idea of the trustworthiness of the distances based on the calculated values of the parameter R , the parallaxes of those stars of the subclasses B8—A5 which are to be found in KAPTEYN's and WEERSMA's¹⁾ list of parallax determinations were compared with those calculated from

$$\pi = \frac{0''.206265}{r} = \frac{0''.206265}{R \cdot 10^{0.2m}}.$$

It is shown that about 60% of the computed parallaxes lie within the mean errors of those measured²⁾.

14. *The velocity distribution.* For all stars brighter than the fifth magnitude and with known proper motion and radial velocity the velocity components were calculated. If W denotes the radial velocity, the components of velocity in the above defined system K are

$$U = Ru', \quad V = Rv', \quad W.$$

From these we get the components in the common astronomical equatorsystem,

$$U'', \quad V'', \quad W''$$

through the relations

$$\begin{aligned} U'' &= \gamma_{11} U + \gamma_{12} V + \gamma_{13} W; \\ V'' &= \gamma_{21} U + \gamma_{22} V + \gamma_{23} W; \\ W'' &= \gamma_{31} U + \gamma_{32} V + \gamma_{33} W; \end{aligned}$$

where γ_{ij} are the direction cosines.

¹⁾ G. P. 24.

²⁾ Compare table III in L. M. I, 76.

The three axes, α_1 , α_2 , α_3 , of the velocity ellipsoid and their directions were then calculated by means of the method given by CHARLIER. For the details I refer to L. M. I, 76. We get the following values of the axes

$$\alpha_1 = 2.71 \pm 0.18;$$

$$\alpha_2 = 5.11 \pm 0.24;$$

$$\alpha_3 = 2.81 \pm 0.11.$$

The following table gives the coordinates of the points toward which the axes are directed.

TABLE III.

THE DIRECTIONS OF THE AXES OF THE VELOCITY ELLIPSOID.

Axis	α	δ	l	b
α_1	349°6	+ 72°0	83°7	+ 11°5
α_2	281°0	— 6°6	354°6	— 4°0
α_3	193°4	+ 17°8	287°0	+ 78°9

l and b are the coordinates in the galactic system used by PICKERING. The north galactic pole has the coordinates

$$\alpha = 190^\circ; \delta = + 28^\circ \text{ (H. A. 56).}$$

The direction of the axis α_2 gives us the *vertex direction* and, accordingly, the coordinates of the *vertex* are

$$\begin{aligned} \alpha &= 281^\circ_0 & l &= 354^\circ_6 \\ \delta &= - 6^\circ_6 & \text{or } b &= - 4^\circ_0. \end{aligned}$$

The velocity ellipsoid seems to be almost an ellipsoid of revolution with its axis of rotation pointing towards the vertex and about twice as great as the other two. Of these α_1 , which lies approximately in the galactic plane, is somewhat greater than α_3 . The direction of the vertex is nearly perpendicular to the center of the system of the B-stars given by CHARLIER¹⁾.

¹⁾ L. M. II, 14.

CHAPTER II.

THE MEAN VALUES OF R^s AND M .

15. In this chapter I will investigate the relations between the mean values of $\frac{1}{R}$, R and M . For this reason let us consider a dominion of the heavens corresponding to a solid angle ω of any form. If the number of stars within this dominion having an apparent magnitude $m \pm \frac{1}{2} dm$ is

$$dm \, a(m),$$

we have according to CHARLIER¹⁾

$$(7) \quad a(m) = \omega \int_0^{\infty} dr \, r^2 D(r) \varphi\left(m - \frac{1}{b} \log r\right),$$

where $D(r)$ is the density of the stars at the distance r from the sun and

$$dM \, \varphi(M)$$

the relative frequency of stars having an absolute magnitude $M \pm \frac{1}{2} dM$.

The equation (7), which is the starting point for our computations in this chapter, was first given—without $D(r)$ —by GYLDÉN 1872 (CHARLIER loc. cit.), in its complete form it was first used by SEELIGER²⁾.

The relation between apparent and absolute magnitude is given from

$$r = 10^{0.2(m-M)} = e^{b(m-M)},$$

where $b = 0.2 \times \text{Mod.} = 0.46054$.

From this we get

$$(8) \quad M = m - 5 \cdot \log r = m - \frac{1}{b} \log r,$$

where $\log r$ is the natural logarithm of r . In theoretical treatments the natural logarithms are to be preferred.

The relation (8) is only valid under the assumption that there exists no absorption of light in space. As yet there is no positive evidence for the existence of any appreciable absorption, and therefore we have no reason to introduce into our research one more unknown function.

¹⁾ *Studies in Stellar Statistics I*, Constitution of the Milky Way. L. M. II, 8, 1912.

²⁾ Betrachtungen über die räumliche Verteilung der Fixsterne. Abhandlungen der K. Bayer. Akademie d. Wiss., II. Cl., XIX. Bd., III. Abt., 1898.

16. We must now make a plausible supposition concerning the form of the function $\varphi(M)$. KAPTEYN¹⁾ has first shown that, for all spectral types together, the best analytical form for the frequency function of the absolute magnitude is a normal error curve (normal curve of type A). This result has later been confirmed by other investigators, and the above form is used in all important researches in stellar statistics.

If we regard stars of a certain spectral type, it lies very near to assume the same form to be valid. Recent investigations made by KAPTEYN²⁾ also support this assumption, at least concerning the spectral types B and A.

In what follows we will therefore assume the function $\varphi(M)$ to have the form

$$(9) \quad \varphi(M) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0)^2}{2\sigma^2}},$$

where M_0 is the mean absolute magnitude and σ the dispersion.

No special assumption regarding the form of the function $D(r)$ need be made for the present.

17. We then start from the following assumptions:

1. *There exists no appreciable absorption of light in space.*
2. *The frequency function of the absolute magnitude has the form (9). This function is independent of the distance.*

18. *The mean value of R^s for stars of a given apparent magnitude m is, with the aid of (7), obtained from*

$$(10) \quad a(m) \overline{R_m^s} = \omega \int_0^\infty dr r^2 D(r) R^s \varphi\left(m - \frac{1}{b} \log r\right).$$

But, from the definition of R , we have

$$R = 10^{-0.2M} = e^{-bM}, \therefore R^s = e^{-sbM}.$$

Now is according to (9)

$$\varphi(M) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0)^2}{2\sigma^2}}.$$

We then have

$$\begin{aligned} R^s \varphi(M) &= e^{-sbM} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0)^2 + 2sb\sigma^2 M}{2\sigma^2}} \\ &= e^{-sbM_0 + \frac{1}{2}s^2b^2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M+sb\sigma^2-M_0)^2}{2\sigma^2}}. \end{aligned}$$

¹⁾ Remarks on the Determination of the Number and Mean Parallax of Stars of different Magnitude and the Absorption of Light in Space. A. J. 24, 115, 1904.

²⁾ On the Parallaxes and Motion of the brighter galactic Helium Stars between galactic Longitudes 150° and 216°. Contributions from the Mount Wilson Solar Observatory, 147, 1918. Ap. J. 47, 1918.

From this we get

$$(11) \quad R^s \varphi(M) = e^{-sbM_0 + \frac{1}{2}s^2b^2\sigma^2} \cdot \varphi(M + sb\sigma^2),$$

or, with the help of (8),

$$(12) \quad R^s \varphi\left(m - \frac{1}{b} \log r\right) = e^{-sbM_0 + \frac{1}{2}s^2b^2\sigma^2} \cdot \varphi\left(m + sb\sigma^2 - \frac{1}{b} \log r\right).$$

Introducing this value of

$$R^s \varphi\left(m - \frac{1}{b} \log r\right)$$

into the equation (10), this equation takes the form

$$a(m) \overline{R_m^s} = e^{-sbM_0 + \frac{1}{2}s^2b^2\sigma^2} \omega \int_0^\infty dr r^2 D(r) \varphi\left(m + sb\sigma^2 - \frac{1}{b} \log r\right).$$

But, according to (7), we have

$$\omega \int_0^\infty dr r^2 D(r) \varphi\left(m + sb\sigma^2 - \frac{1}{b} \log r\right) = a(m + sb\sigma^2),$$

from which we get the wanted mean value in the form

$$(13) \quad \overline{R_m^s} = e^{-sbM_0 + \frac{1}{2}s^2b^2\sigma^2} \frac{a(m + sb\sigma^2)}{a(m)}.$$

19. *The mean value of R^s for stars brighter than a given apparent magnitude m is denoted by*

$$\overline{R}_{-\infty, m}^s.$$

From (7) we then get

$$(14) \quad \int_{-\infty}^m dm a(m) \overline{R}_{-\infty, m}^s = \omega \int_0^\infty dr r^2 D(r) \int_{-\infty}^m dm R^s \varphi\left(m - \frac{1}{b} \log r\right)$$

or with the help of the relation (12),

$$= e^{-sbM_0 + \frac{1}{2}s^2b^2\sigma^2} \omega \int_0^\infty dr r^2 D(r) \int_{-\infty}^m dm \varphi\left(m + sb\sigma^2 - \frac{1}{b} \log r\right).$$

But

$$(14^*) \quad \omega \int_0^\infty dr r^2 D(r) \int_{-\infty}^m dm \varphi\left(m + sb\sigma^2 - \frac{1}{b} \log r\right) = \omega \int_0^\infty dr r^2 D(r) \int_{-\infty}^{m+sb\sigma^2} dm \varphi\left(m - \frac{1}{b} \log r\right).$$

Let us now put the number of stars brighter than the apparent magnitude m equal to $A(m)$, for which we get the following expression

$$(15) \quad A(m) = \int_{-\infty}^m dm a(m) = \omega \int_0^{\infty} dr r^3 D(r) \int_{-\infty}^m dm \varphi\left(m - \frac{1}{b} \log r\right).$$

According to (15) we thus get

$$\omega \int_0^{\infty} dr r^3 D(r) \int_{-\infty}^{m + s b \sigma^2} dm \varphi\left(m - \frac{1}{b} \log r\right) = A(m + s b \sigma^2),$$

and we then find by means of this relation and (14*) from (14) the mean value in the form

$$(16) \quad \overline{R^s}_{-\infty, m} = e^{-s b M_0 + \frac{1}{2} s^2 b^2 \sigma^2} \frac{A(m + s b \sigma^2)}{A(m)}.$$

The mean value of R^s for stars brighter than m_2 but fainter than m_1 we get in an analogous manner from

$$(16^*) \quad \overline{R^s}_{m_1, m_2} = e^{-s b M_0 + \frac{1}{2} s^2 b^2 \sigma^2} \frac{A(m_2 + s b \sigma^2) - A(m_1 + s b \sigma^2)}{A(m_2) - A(m_1)}.$$

20. For $s = -1$ the relation (16) gives us the mean value of $\frac{1}{R}$ for all stars brighter than a given apparent magnitude. We get

$$(17) \quad \left(\frac{1}{R}\right)_{-\infty, m} = e^{b M_0 + \frac{1}{2} b^2 \sigma^2} \frac{A(m - b \sigma^2)}{A(m)}.$$

We may call $\frac{1}{R}$ the *reduced parallax* of a star, and the corresponding mean value the *mean reduced parallax*. Now by means of the method developed in the first chapter we just determine the mean reduced parallax for stars brighter than a given apparent magnitude or between two given limits of magnitude. Therefore the relation (17) is of special interest for us.

If the absolute magnitude is a constant ($\sigma = 0$; $M = M_0$), the mean reduced parallax is also a constant for any value of m .

But, on the other hand, if the mean reduced parallax is a constant for different values of m , we cannot at once conclude that the absolute magnitude must be a constant too. For the mean reduced parallax is constant, if the ratio

$$\frac{A(m - b \sigma^2)}{A(m)}$$

is independent of m , which is the case if $A(m)$ has the form

$$(18) \quad A(m) = C e^{k m}.$$

This form of $A(m)$ has been found by SEELIGER¹⁾ to be approximately valid for the stars down to the ninth magnitude if all spectral types are treated together. He has shown, too, that from this form of $A(m)$ follows, independent of the form of the frequency function of the absolute magnitude—provided this function is independent of the distance—a density function of the form of

$$(18^*) \quad D(r) = \gamma \cdot r^{-s_1}.$$

The relation between the two constants k and s_1 is of the form

$$k = b(3 - s_1). \quad (b = 0.46054).$$

If $s_1 = 0$ we get the case of constant density where

$$(19) \quad A(m) = Ce^{3bm}.$$

21. Now let us consider the equation (17) a little closer. The value of $\left(\frac{1}{R}\right)_{-\infty, m}$ is determined from the reduced proper motions of the stars, the function $A(m)$ up to the limiting magnitude is known with sufficient accuracy. The equation in question thus gives us a relation between M_0 and σ that is between the mean value of the absolute magnitude and the dispersion.

If $\frac{A(m - b\sigma^2)}{A(m)}$ is a function of m , the mean reduced parallax must vary with m .

If we determine the mean reduced parallax for stars brighter than m_1 and for stars brighter than m_2 , we get the two equations

$$(20) \quad \begin{aligned} \left(\frac{1}{R}\right)_{-\infty, m_1} &= e^{bM_0 + \frac{1}{2}b^2\sigma^2} \frac{A(m_1 - b\sigma^2)}{A(m_1)}, \\ \left(\frac{1}{R}\right)_{-\infty, m_2} &= e^{bM_0 + \frac{1}{2}b^2\sigma^2} \frac{A(m_2 - b\sigma^2)}{A(m_2)}. \end{aligned}$$

If we divide these two equations, we get

$$(21) \quad \frac{A(m_1 - b\sigma^2)}{A(m_2 - b\sigma^2)} = \frac{\left(\frac{1}{R}\right)_{-\infty, m_1}}{\left(\frac{1}{R}\right)_{-\infty, m_2}} \cdot \frac{A(m_1)}{A(m_2)},$$

where the term on the right hand side is known. Now if the variation in the mean reduced parallax is great enough, we may determine σ^2 from (21) either by trials or by forming from the observations a suitable approximate expression of $A(m)$ as function of m and then solve σ^2 .

When σ is known, we get the value of M_0 from one of the equations (20). We have thus—at least theoretically—a possibility to determine from the reduced proper

¹⁾ Loc. cit.

motions the values of the two constants of the frequency function of the absolute magnitude of the stars in question *without making any supposition concerning the form of the density function*.

In the next chapter we will try to apply this method to the material which is at our disposal.

22. If the constants M_0 and σ are known, we may from the distribution of the apparent magnitudes determine the mean values of M for stars of a given apparent magnitude and stars brighter than a given apparent magnitude. These mean values are of course not identical with M_0 . Then the material upon which all our researches are based consists of stars up to a given apparent magnitude. And such a material is naturally very select, as the absolutely bright stars are richly represented at the expense of the absolutely faint ones.

23. The mean value of M for a given apparent magnitude m is computed in the following way. In equation (7) the function $a(m)$ is defined through an integral which contains m as a parameter. We now make the supposition concerning $D(r)$ that this function is thus conditioned that the integral (7), which defines $a(m)$, fulfils the conditions necessary to permit differentiation with respect to the parameter m^1 .

We have according to (7)

$$(7) \quad a(m) = \omega \int_0^{\infty} dr r^2 D(r) \varphi \left(m - \frac{1}{b} \log r \right).$$

From this we get the mean value in question from

$$a(m) \overline{M}_m = \omega \int_0^{\infty} dr r^2 D(r) M \varphi \left(m - \frac{1}{b} \log r \right)$$

or, with regard to (8),

$$(22) \quad a(m) \overline{M}_m = \omega \int_0^{\infty} dr r^2 D(r) \left(m - \frac{1}{b} \log r \right) \varphi \left(m - \frac{1}{b} \log r \right).$$

But differentiating (7) with respect to m , we get, observing that $\varphi(M)$ has the form (9),

$$\begin{aligned} \frac{da(m)}{dm} &= -\frac{\omega}{\sigma^2} \int_0^{\infty} dr r^2 D(r) \left(m - \frac{1}{b} \log r \right) \varphi \left(m - \frac{1}{b} \log r \right) + \\ &+ \frac{\omega M_0}{\sigma^2} \int_0^{\infty} dr r^2 D(r) \varphi \left(m - \frac{1}{b} \log r \right), \end{aligned}$$

from which relation by means of (7) we get from (22)

$$a(m) \overline{M}_m = a(m) M_0 - \sigma^2 \frac{da(m)}{dm}$$

¹⁾ Compare for inst. SERRET-SCHEFFERS, Lehrbuch der Differential- und Integralrechnung, II, 5. Aufl., page 184.

and hence

$$(23) \quad \bar{M}_m = M_0 - \frac{\sigma^2}{a(m)} \frac{d a(m)}{d m}$$

or

$$(24) \quad \bar{M}_m = M_0 - \sigma^2 \frac{d(\log a(m))}{d m}.$$

In the same manner we may also calculate the moments of higher order. For the dispersion in the absolute magnitude for a given m , for instance, we get

$$(24^*) \quad \sigma^2_m = \sigma^2 + \sigma^4 \frac{d^2(\log a(m))}{d m^2}.$$

24. *The mean value of M for stars brighter than a given apparent magnitude m_1 is calculated in an analogous manner.*

We have from (7)

$$(24a) \quad \begin{aligned} \int_{-\infty}^{m_1} d m a(m) \bar{M}_{-\infty, m_1} &= \omega \int_0^{\infty} d r r^2 D(r) \int_{-\infty}^{m_1} d m \left(m - \frac{1}{b} \log r \right) \varphi \left(m - \frac{1}{b} \log r \right) = \\ &= M_0 \omega \int_0^{\infty} d r r^2 D(r) \int_{-\infty}^{m_1} d m \varphi \left(m - \frac{1}{b} \log r \right) + \\ &\quad + \omega \int_0^{\infty} d r r^2 D(r) \int_{-\infty}^{m_1} d m \left(m - \frac{1}{b} \log r - M_0 \right) \varphi \left(m - \frac{1}{b} \log r \right). \end{aligned}$$

But now we have, according to the form (9) of $\varphi(M)$

$$\int_{-\infty}^{m_1} d m \left(m - \frac{1}{b} \log r - M_0 \right) \varphi \left(m - \frac{1}{b} \log r \right) = -\sigma^2 \varphi \left(m_1 - \frac{1}{b} \log r \right)$$

and hence follows from (24a)

$$A(m_1) \bar{M}_{-\infty, m_1} = A(m_1) M_0 - a(m_1) \sigma^2,$$

from which we get

$$\bar{M}_{-\infty, m_1} = M_0 - \sigma^2 \frac{a(m_1)}{A(m_1)}.$$

But $a(m) = \frac{d(A(m))}{d m}$, so that we may write the above relation in the form of

$$(24b) \quad \bar{M}_{-\infty, m_1} = M_0 - \sigma^2 \left\{ \frac{d(\log A(m))}{d m} \right\}_{m=m_1}$$

analogous with (24).

If we suppose the constants M_0 and σ to be the same over the whole heavens, the formulae deduced in this chapter are valid for the whole sky, independent of the variation of the density function in different parts.

25. We shall now look at the different mean values under the most usual assumptions concerning $a(m)$.

I.
$$a(m) = C e^{b(3-s_1)m},$$

which also includes the case of constant density ($s_1 = 0$).

From (13) and (16) we get for $s = 1$

$$\bar{R}_m = \bar{R}_{-\infty, m} = e^{-bM_0 + b^2\sigma^2\left(\frac{7}{2} - s_1\right)}$$

and for $s = -1$

$$\left(\frac{1}{R}\right)_m = \left(\frac{1}{R}\right)_{-\infty, m} = e^{bM_0 - b^2\sigma^2\left(\frac{5}{2} - s_1\right)}.$$

Hence it follows that

$$\bar{R}_m \cdot \left(\frac{1}{R}\right)_m = e^{b^2\sigma^2};$$

Further we find

$$\bar{M}_m = \bar{M}_{-\infty, m} = M_0 - b(3 - s_1)\sigma^2$$

Is the density a constant, then

$$\bar{M}_m = \bar{M}_{-\infty, m} = M_0 - 3b\sigma^2 = M_0 - 1.382\sigma^2.$$

This relation is given by EDDINGTON in *Stellar Movements*, page 172, but it is not clearly pointed out there that the relation is only valid under the assumption of constant density.

When the apparent magnitudes follow the above form, we find that the mean absolute magnitudes for stars of a given apparent magnitude and for stars brighter than a given apparent magnitude are equal, and $(3 - s_1)b\sigma^2$ magnitudes brighter than M_0 .

Is the density a constant, we get

$$\begin{aligned} \text{for } \sigma = 1 & \dots\dots \bar{M}_m = M_0 - 1^m_4, \\ \sigma = 2 & \dots\dots \bar{M}_m = M_0 - 5^m_5, \\ \sigma = 3 & \dots\dots \bar{M}_m = M_0 - 12^m_4. \end{aligned}$$

The difference between M_0 and $\bar{M}_m$ increases rapidly with the dispersion.

For stars of all spectral types together the value of σ is about 3 magnitudes. If the density were a constant, the mean absolute magnitude of those stars which form the material for our investigations would be about 12 magnitudes brighter than M_0 . Now the density is not constant, but if we continue to consider all spectral types, we have for the brighter stars approximately

$$a(m) = C e^{2bm}$$

and also

$$\bar{M}_m = M_0 - 8^m_3 \text{ for } \sigma = 3.$$

I have given these figures in order to show how select a material are the stars brighter than a given apparent magnitude.

II. The counts of stars down to the 17th magnitude by CHAPMAN and MELOTTE¹⁾ show that $a(m)$ very well follows a normal curve, that is

$$a(m) = \frac{N}{\alpha\sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\alpha^2}}.$$

With this form of $a(m)$ we get

$$\begin{aligned} \bar{R}_m &= e^{-bM_0 - b\frac{\sigma^2}{\alpha^2}(m-m_0) + \frac{1}{2}b^2\frac{\sigma^2}{\alpha^2}(\alpha^2 - \sigma^2)}, \\ \left(\frac{1}{R}\right)_m &= e^{+bM_0 + b\frac{\sigma^2}{\alpha^2}(m-m_0) + \frac{1}{2}b^2\frac{\sigma^2}{\alpha^2}(\alpha^2 - \sigma^2)}. \end{aligned}$$

If, according to CHARLIER²⁾, we put

$$1 - \frac{\sigma^2}{\alpha^2} = \lambda_1,$$

we have

$$\bar{R}_m \cdot \left(\frac{1}{R}\right)_m = e^{b^2\sigma^2\lambda_1}$$

Further is

$$\bar{M}_m = M_0 + \frac{\sigma^2}{\alpha^2}(m - m_0).$$

For $m < m_0$ we have $\bar{M}_m < M_0$,

» $m = m_0$ » » $\bar{M}_m = M_0$

and » $m > m_0$ » » $\bar{M}_m > M_0$.

When the apparent magnitude is increased by one magnitude, the corresponding increase for the mean absolute magnitude is $\frac{\sigma^2}{\alpha^2}$ magnitudes.

26. From the relation (1*)

$$r = R \cdot 10^{0.2m} = R e^{bm}$$

we get

$$\frac{1}{r} = \frac{1}{R} e^{-bm}.$$

For stars of a given apparent magnitude we thus have

$$\left(\frac{1}{r}\right)_m = \left(\frac{1}{R}\right)_m e^{-bm}$$

¹⁾ The Number of Stars of each Photographic Magnitude down to 17th in different Galactic Latitudes. Memoirs of the Royal Astronomical Society, vol. LX, part IV, 1914.

²⁾ L. M. II, 8.

or, by means of (13)

$$(25) \quad \left(\frac{1}{r}\right)_m = e^{-b(m-M_0) + \frac{1}{2}b^2\sigma^2} \frac{a(m-b\sigma^2)}{a(m)}.$$

Introducing in stead of the distance the parallax through the relation

$$\pi = \frac{0''.206265}{r} \quad (r \text{ given in siriometers})$$

we get the mean parallax for stars of a given apparent magnitude

$$(26) \quad \bar{\pi}_m = 0''.206265 e^{-b(m-M_0) + \frac{1}{2}b^2\sigma^2} \frac{a(m-b\sigma^2)}{a(m)}.$$

This form for $\bar{\pi}_m$ is valid under the assumptions made in § 17.

Now KAPTEYN and others have found that the mean parallaxes determined from the parallactic motion may be represented by

$$(27) \quad {}^{10}\log \bar{\pi}_m = a_1 - a_2 m.$$

Recently KAPTEYN, VAN RHIJN and WEERSMA¹⁾, by an investigation of the mean parallax from proper motions of stars down to the ninth magnitude for each spectral type separately, have found the same formula (27) to be valid. The constant a_1 in this formula is found to vary with the spectral type, but a_2 seems to be equal for all spectral types.

From (27) follows

$$(28) \quad \bar{\pi}_m = 10^{a_1 - a_2 m} = e^{5 b a_1 - 5 b a_2 m}.$$

If we compare this form with (26) we find that the conditions under which (26) takes the same form as (28) are the following.

1.

$$\frac{a(m-b\sigma^2)}{a(m)} \text{ independent of } m,$$

from which follows, as will be demonstrated in the following paragraph, that $a(m)$ must have the form of

$$(29) \quad a(m) = C e^{km}.$$

Then (26) passes into

$$(29^*) \quad \bar{\pi}_m = 0''.206265 e^{-b(m-M_0) + b\sigma^2\left(\frac{b}{2} - k\right)},$$

from which, putting $0.206265 = e^{-3.43 b}$ and comparing with (28), we get

$$(29^{**}) \quad \begin{aligned} a_1 &= 0.2 \left(M_0 - 3.43 + \sigma^2 \left[\frac{b}{2} - k \right] \right); \\ a_2 &= 0.2. \end{aligned}$$

¹⁾ The Secular Parallax of the Stars of different Magnitude, Galactic Latitude and Spectrum. G. P. 29, 1918.

2.

$$\frac{a(m - b\sigma^2)}{a(m)} = C \cdot e^{a_1 m},$$

from which follows (see the following paragraph).

$$(30) \quad a(m) = e^{a m^2 + b m + c} = \frac{N}{a\sqrt{2\pi}} e^{-\frac{(m - m_0)^2}{2a^2}} \quad (\text{when } a \text{ is negative}),$$

(26) then takes the form of

$$(30^*) \quad \bar{\pi}_m = 0''_{.206265} e^{-b(m - M_0) + b \frac{\sigma^2}{a^2} (m - m_0) + \frac{1}{2} b^2 \frac{\sigma^2}{a^2} (a^2 - \sigma^2)}$$

and hence, putting

$$1 - \frac{\sigma^2}{a^2} = \lambda_1.$$

$$(30^{**}) \quad \begin{aligned} a_1 &= 0.2 \left(M_0 - 3.43 - m_0(1 - \lambda_1) + \frac{1}{2} b \sigma^2 \lambda_1 \right); \\ a_2 &= 0.2 \lambda_1; \end{aligned}$$

as λ_1 must always be smaller than unity, we have $a_2 < 0.2$.

The value of a_2 has been found in all researches to be smaller than 0.2, the form of the mean parallax thus points in the direction that $a(m)$ —at least approximately—has the form (30), which is also confirmed as far as our observations reach.

The result is derived under the assumptions made in § 17. From these assumptions and the form (30) for $a(m)$ follows the density function derived by SCHWARZSCHILD¹⁾. With this density function follows the form (29) of $\bar{\pi}_m$. The relations (30^{**}) between the constants are given by CHARLIER²⁾.

27. In the preceding paragraph we had to determine $a(m)$ as a function of m from the relation

$$(31) \quad \frac{a(m - b\sigma^2)}{a(m)} = e^{a m + \beta}.$$

Putting $b\sigma^2 = h$ and taking the logarithms we get

$$(31^*) \quad f(m) - f(m - h) = -a m - \beta,$$

where

$$(31^{**}) \quad f(m) = \log a(m).$$

The relation (31^{*}) is a difference equation in its most simple form. As is well known from the theory of difference equations, the principal solution of the equation (31^{*}) is

$$(32) \quad f(m) = a m^2 + b m + c.$$

¹⁾ Über die Integralgleichungen der Stellarstatistik. A. N. 185, 81, 1910.

²⁾ L. M. II, 8.

The general solution is then of the form

$$(32^*) \quad f(m) = am^2 + bm + c + \omega_h(m),$$

where $\omega_h(m)$ is any periodic function of m with a period equal to h .

From (31**) follows that the general solution of (31) has the form

$$a(m) = e^{am^2 + bm + c + \omega_h(m)}.$$

But according to the definition of $a(m)$ the only available solution is the principal one, and hence it follows

$$a(m) = e^{am^2 + bm + c}.$$

From (31) we find the following relations between the constants:—

$$\begin{aligned} \alpha &= -2ah, \\ \beta &= ah^2 - bh. \end{aligned}$$

As the constant a_2 , defined in § 26, is found to be less than 0.2, we shall have α positive, from which follows that a is negative.

We may thus write $a(m)$ in the form of

$$a(m) = \frac{N}{\alpha\sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2a^2}}.$$

The form (29) is only a special case when $\alpha = 0$, from which follows that a too is equal to 0.

CHAPTER III.

A. THE BOSS STARS.

28. We shall now from the points of view of the preceding chapter treat the BOSS stars more thoroughly. As the galactic plane is the natural plane of symmetry in our stellar system, I have in what follows assigned these stars to a galactic system of coordinates. The location of the galactic plane differs a little according to different investigators, I will here use the galactic system defined by PICKERING. In this system, the north galactic pole has the coordinates

$$\alpha = 190^\circ, \quad \delta = +28^\circ.$$

Towards this point our positive axis of z is directed. The two other axes—in the galactic plane—are chosen so that

$$\begin{array}{llll} \text{the positive axis of } X & \text{is directed towards } \alpha = 280^\circ, & \delta = 0^\circ, \\ \text{» } Y & \text{» } \alpha = 10^\circ, & \delta = +62^\circ. \end{array}$$

The direction cosines relating the axes of this new system (K_g) with those of the common astronomical equator system (K'') are given in the following scheme:—

	K''		
	X''	Y''	Z''
X_g	ϵ_{11}	ϵ_{21}	ϵ_{31}
$K_g: Y_g$	ϵ_{12}	ϵ_{22}	ϵ_{32}
Z_g	ϵ_{13}	ϵ_{23}	ϵ_{33}

where

$$\begin{array}{lll} \epsilon_{11} = +0.1786, & \epsilon_{21} = -0.9848, & \epsilon_{31} = 0, \\ \epsilon_{12} = +0.4623, & \epsilon_{22} = +0.0815, & \epsilon_{32} = +0.8829, \\ \epsilon_{13} = -0.8695, & \epsilon_{23} = -0.1533, & \epsilon_{33} = +0.4695. \end{array}$$

The components u' and v' of the reduced proper motions used in the first chapter were referred to a system of coordinates denoted by K . Through a rotation around the axis of Z in this system we get the corresponding components u_0' and v_0' in a new system of coordinates K_0 . The axis of Z in this system thus coincides with the corresponding axis in the system K , *viz.* directed along the radius vector to the star, the axis of X lies in the direction of increasing galactic longitude, the axis of Y in the direction of increasing galactic latitude. The following scheme gives the direction cosines between the axes of this new system K_0 and those of the above defined galactic system K_g .

	$K_g:$		
	X_g	Y_g	Z_g
X_0	β_{11}	β_{21}	β_{31}
$K_0: Y_0$	β_{12}	β_{22}	β_{32}
Z_0	β_{13}	β_{23}	β_{33}

where

$$\begin{aligned}\beta_{11} &= -\sin l; & \beta_{21} &= +\cos l; & \beta_{31} &= 0; \\ \beta_{12} &= -\sin b \cos l; & \beta_{22} &= -\sin b \sin l; & \beta_{32} &= +\cos b; \\ \beta_{13} &= +\cos b \cos l; & \beta_{23} &= +\cos b \sin l; & \beta_{33} &= +\sin b;\end{aligned}$$

l and b are the galactic longitude and latitude of a star.

Thus the direction cosines β_{ij} are the same functions of l and b as the direction cosines γ_{ij} , relating the systems K'' and K , of α and δ .

29. In order to facilitate the computations I have not, as in the first chapter, formed the equations for each individual star, but made use of the simplification introduced by CHARLIER. He divides the sky into 48 "squares" of equal area. In L. M. II, 8 this division was made with respect to the equator, but later on CHARLIER has made an analogous division with the galactic plane as plane of symmetry. This later division is used here. The zone between the galactic equator and the parallel $b = +30^\circ$ is divided into 12 squares $GC_1 - GC_{12}$, GC_1 containing those stars whose l lies between 0° and 30° , GC_2 those with l between 30° and 60° and so on. The zone between $b = +30^\circ$ and $b = +66^\circ 26' 6''$ is divided into 10 squares $GB_1 - GB_{10}$, and the calotte around the north galactic pole into two, GA_1 and GA_2 . The southern galactic hemisphere is divided in the same manner, 12 squares $GD_1 - GD_{12}$ between $b = 0^\circ$ and $b = -30^\circ$, 10 squares $GE_1 - GE_{10}$ between $b = -30^\circ$ and $b = -66^\circ 26' 6''$ and two, GF_1 and GF_2 around the south galactic pole.

Now let U_g, V_g, W_g denote the linear velocity components of a star with respect to the sun in the galactic System K_g . We then have

$$(34) \quad \begin{aligned}u_0' &= \beta_{11} \frac{U_g}{R} + \beta_{21} \frac{V_g}{R}, \\ v_0' &= \beta_{12} \frac{U_g}{R} + \beta_{22} \frac{V_g}{R} + \beta_{32} \frac{W_g}{R}\end{aligned}$$

analogous with equations (4) in the first chapter.

Instead of calculating the coefficients β_{ij} for each star separately we refer all the stars in a square to the centre of gravity of this square, thus letting all stars have the same values of β_{ij} as the centre of gravity. As the coordinates in l and b of the centre in a certain galactic square have the same values as the coordinates in α and δ of the corresponding equatorial square, the direction cosines β_{ij} have the same numerical values as the corresponding γ_{ij} for the equatorial squares. Therefore we can use also for β_{ij} the tables for γ_{ij} given by WICKSELL¹⁾ and GYLLENBERG²⁾.

¹⁾ The General Characteristics of the Frequency Function of Stellar Movements as derived from the Proper Motions of the Stars. L. M. II, 12, 1915.

²⁾ L. M. II, 13.

If we then form the mean values of the reduced proper motions u_0' and v_0' for each square, viz.

$$\bar{u}_0' = \frac{\sum u_0'}{n}, \quad \bar{v}_0' = \frac{\sum v_0'}{n}$$

where n is the number of stars in the square in question, we get with the help of the equations (34) for each square the equations

$$(34^*) \quad \begin{aligned} \bar{u}_0' &= \beta_{11} \left(\frac{\bar{U}_g}{R} \right) + \beta_{21} \left(\frac{\bar{V}_g}{R} \right), \\ \bar{v}_0' &= \beta_{12} \left(\frac{\bar{U}_g}{R} \right) + \beta_{22} \left(\frac{\bar{V}_g}{R} \right) + \beta_{32} \left(\frac{\bar{W}_g}{R} \right). \end{aligned}$$

When forming the mean values $\bar{u}_0'$ and $\bar{v}_0'$ I have in conformity with WICKSELL¹⁾ taken together diametrically situated squares and then changed the sign of the component u_0' for stars in the southern galactic hemisphere. We thus get two equations of the form (34*) for each such double-square. From these systems of equations we form the corresponding normal equations, which naturally are similar to those given in (4*). Adding the corresponding normal equations received from the motions in galactic longitude and latitude we get a system of equations, analogous to (5), to solve our unknown quantities

$$\left(\frac{\bar{U}_g}{R} \right), \left(\frac{\bar{V}_g}{R} \right) \text{ and } \left(\frac{\bar{W}_g}{R} \right).$$

We now suppose $\frac{1}{R}$ to be independent of the velocities, so that we have

$$\left(\frac{\bar{U}_g}{R} \right) = \left(\frac{1}{R} \right) \bar{U}_g; \quad \left(\frac{\bar{V}_g}{R} \right) = \left(\frac{1}{R} \right) \bar{V}_g; \quad \left(\frac{\bar{W}_g}{R} \right) = \left(\frac{1}{R} \right) \bar{W}_g.$$

But if S is the velocity of the sun, we have

$$S = \sqrt{\bar{U}_g^2 + \bar{V}_g^2 + \bar{W}_g^2},$$

from which follows

$$S \left(\frac{1}{R} \right) = \sqrt{\left(\frac{\bar{U}_g}{R} \right)^2 + \left(\frac{\bar{V}_g}{R} \right)^2 + \left(\frac{\bar{W}_g}{R} \right)^2}.$$

From this relation we get, with the help of the known value of the velocity of the sun, the value of $\left(\frac{1}{R} \right)$, the mean reduced parallax.

The method of solving the normal equations and calculating the mean errors is given in the first chapter.

30. When forming the mean values $\bar{u}_0'$ and $\bar{v}_0'$ for each double-square we meet with the same difficulty as when treating the corresponding problem concerning the common proper motions. Eventual large values of u_0' and v_0' will influence the mean

¹⁾ Loc. cit.

values very decisively, one large motion can completely dislocate the means of a certain square. It is therefore a very difficult thing to decide whether those large motions are to be excluded or not when forming the mean values, and in the former case to what extent the exclusion is to be made. Happily for the A-stars the large motions are few in number and of a less dangerous quality, therefore I have in the investigation made in the first chapter excluded only one single star.

In what follows I have treated together the subclasses B8—A5 partly in order to get a greater material, partly owing to the facts mentioned in § 12. I have not now excluded any star on account of large proper motion. I consider this correct for the material in question, at least as long as the question is only to determine the mean reduced parallax. The errors in this mean value caused by the large motions, must be of a more accidental character, whereas any exclusion of large motions will easily cause errors of *systematical* character. If, therefore, I have made no exclusion on account of large proper motion, I have, on the other hand, excluded other stars, which would cause systematical errors, namely those belonging to local star-streams. A great many of the stars belonging to the magnificent moving clusters of Taurus and Ursa Major are included in the material for this investigation. As belonging to the Taurus-stream I have excluded all stars of the spectral types in question given by L. BOSS¹⁾ except the most doubtful star Boss 1051, in all 26 stars. To the Ursa Major cluster I have reckoned those stars given by HERTZSPRUNG²⁾, and thus 12 stars of spectral types A—A5 are excluded. For the same reason I have in the case of physically connected double stars excluded one of the components; by computing the reduced proper motions I have in these cases reckoned with a value of m corresponding to the mean value of the apparent magnitudes of the two components.

31. The material thus reduced consists of 1434 stars with known proper motion brighter than the sixth magnitude. This material I have divided into two groups. The first group includes all stars at a distance greater than 30° from the galactic plane, also embracing the squares GA_1 , GA_2 , GB_1 — GB_{10} , GE_1 — GE_{10} , GF_1 , GF_2 , in all 524 stars. The second group includes the other stars, also those situated between $b = +30^\circ$ and $b = -30^\circ$. Their number is 910.

Treating each group separately according to the given method we get the following results.

GROUP I. NON-GALACTIC STARS BRIGHTER THAN THE SIXTH MAGNITUDE.

$$N = 524.$$

$$(35) \quad \begin{aligned} \left(\frac{\overline{U_g}}{R} \right) &= -2.221 \pm 0.195, \\ \left(\frac{\overline{V_g}}{R} \right) &= -0.688 \pm 0.196, \\ \left(\frac{\overline{W_g}}{R} \right) &= -0.990 \pm 0.268. \end{aligned}$$

¹⁾ Convergent of a Moving Cluster in Taurus. A. J. 26, 31, 1908.

²⁾ On new Members of the System of the Stars β , γ , δ , ϵ , ζ Ursae Majoris. Ap. J. 30, 135, 1909.

From this we get

$$S\left(\frac{1}{R}\right) = 2.513 \pm 0.208.$$

The unknown quantities are given in radians per stellar year.

Using the same value of the velocity of the sun as in the first chapter,

$$S = 4.167 \pm 0.305 \text{ sir./st.},$$

we get

$$(35^*) \quad \left(\frac{1}{R}\right) = 0.608 \pm 0.067,$$

$$R = 1.66, \quad M = -1.10.$$

The values of R and M are given for comparison with the corresponding values in table I.

They are computed by means of the relations

$$R = \frac{1}{\left(\frac{1}{R}\right)}; \quad M = 5 \cdot {}^{10}\log \left(\frac{1}{R}\right).$$

$\left(\frac{1}{R}\right)$ is the mean reduced parallax for stars brighter than the sixth magnitude, also that same mean value, which in the second chapter was denoted by

$$\left(\frac{1}{R}\right)_{-\infty, m}.$$

In the following I have omitted the indices and used the more simple denomination of this mean value.

From (35) we get with the help of a system of equations analogous to (6*) the galactic longitude (L) and latitude (B) of the apex. From L and B the corresponding values of A and D are calculated.

We get

$$\begin{aligned} L &= 16^\circ_0 \pm 5^\circ_1, & B &= +23^\circ_2 \pm 4^\circ_7, \\ A &= 265^\circ_4, & D &= +24^\circ_0. \end{aligned}$$

The mean errors in B and L are calculated with the help of the formulas

$$\varepsilon(B) = \frac{\varepsilon \left[S\left(\frac{1}{R}\right) \right]}{S\left(\frac{1}{R}\right)}, \quad \varepsilon(L) = \frac{\varepsilon(B)}{\cos B};$$

which have been given by CHARLIER¹⁾.

¹⁾ L. M. II, 9, page 74.

GROUP II. GALACTIC STARS BRIGHTER THAN THE SIXTH MAGNITUDE. $N = 910$.

$$(36) \quad \begin{aligned} \left(\frac{\bar{U}_g}{R}\right) &= -2.051 \pm 0.119, \\ \left(\frac{\bar{V}_g}{R}\right) &= -0.707 \pm 0.119, \\ \left(\frac{\bar{W}_g}{R}\right) &= -0.960 \pm 0.090, \\ S\left(\frac{1}{R}\right) &= 2.372 \pm 0.115. \end{aligned}$$

$$(36^*) \quad \left(\frac{1}{R}\right) = 0.569 \pm 0.050.$$

$$\begin{aligned} R &= 1.76, & M &= -1.22, \\ L &= 19^\circ_0 \pm 3^\circ_1, & B &= +23^\circ_9 \pm 2^\circ_8, \\ A &= 265^\circ_9, & D &= +27^\circ_0. \end{aligned}$$

We have thus got a somewhat larger value of $\left(\frac{1}{R}\right)$ for group I than for group II, but with regard to the mean errors we cannot consider the difference significant.

The variation with galactic latitude then seems to be inconsiderable. The coordinates found for the apex agree well for the two groups. In the solution of the necessary normal equations we have given equal weight to each square, which is justified because the number of stars per square does not vary very much within the same group. With this method of weighting, the normal equations take a very simple form.

We shall now combine the two groups to get a definite value of the mean reduced parallax, making allowance for the fact that the galactic squares have by far a greater number of stars than the non-galactic. We therefore give each galactic square the weight 1, each non-galactic the weight $\frac{524}{910}$, equal to the proportion of stars in the resp. groups.

We thus get the following results.

GROUP III. ALL STARS BRIGHTER THAN THE SIXTH MAGNITUDE. $N = 1434$.

$$(37) \quad \begin{aligned} \left(\frac{\bar{U}_g}{R}\right) &= -2.129 \pm 0.114, \\ \left(\frac{\bar{V}_g}{R}\right) &= -0.673 \pm 0.114, \\ \left(\frac{\bar{W}_g}{R}\right) &= -0.966 \pm 0.104, \\ S\left(\frac{1}{R}\right) &= 2.483 \pm 0.112. \end{aligned}$$

$$(37^*) \quad \left(\frac{1}{R}\right) = 0.584 \pm 0.051.$$

$$\begin{aligned} R &= 1.71, & M &= -1.17, \\ L &= 17^\circ_5 \pm 2^\circ_8, & B &= +23^\circ_4 \pm 2^\circ_6, \\ A &= 265^\circ_8, & D &= +25^\circ_5. \end{aligned}$$

32. By the above computations of the mean reduced parallax we have not, on account of the arguments given in § 30, excluded any star on account of large proper motion. However, to study the effect of a relatively strong exclusion on the mean value in question, I have also performed the corresponding computation after an exclusion according to the following principles. The proper motion (common or reduced) of a star is composed of two components, one arising from the parallactic motion and thus for its magnitude dependent on the position with regard to the apex, the other originating in the peculiar motion of the star and dependent on the position with regard to the vertex. Therefore it cannot be considered right only to put an upper limit for the proper motion; when deciding whether a proper motion is to be considered as large or not we ought to have regard to the location of the star in question. The best thing to do would be to examine each square separately, but as the number of stars in each square is relatively small, the mean errors in the characteristics are large and on account of this a certain arbitrariness is introduced. To avoid this disadvantage I have, when determining which stars were to be excluded, put together those squares which are symmetrically situated with respect to the vertex, taking this to be situated midway between the squares C_{12} and D_{12} .

We thus get the following groups for the galactic squares

- (a) $GC_6, GC_{12}, GD_6, GD_{12}$,
- (b) $GC_1, GC_5, GC_7, GC_{11}, GD_1, GD_5, GD_7, GD_{11}$,
- (c) $GC_2, GC_4, GC_8, GC_{10}, GD_2, GD_4, GD_8, GD_{10}$,
- (d) GC_3, GC_9, GD_3, GD_9 ,

and for the non-galactic squares

- (e) $GB_1, GB_4, GB_5, GB_6, GB_9, GB_{10}, GE_1, GE_4, GE_5, GE_6, GE_9, GE_{10}$,
- (f) $GA_1, GA_2, GB_2, GB_3, GB_7, GB_8, GE_2, GE_3, GE_7, GE_8, GF_1, GF_2$.

In each of these six groups the weighted means of the dispersions in u_0' and v_0' are calculated and those stars are excluded whose u_0' or v_0' differ from the resp. means of u_0' and v_0' more than 3 times the dispersion. The means in question varying from square to square within each group, I have used those obtained when putting together diametrically situated squares.

Applying this method of exclusion to the present material we find that 48 stars ought to be excluded, 15 in the non-galactic group and 33 in the galactic region.

From this reduced material we get the following results.

GROUP I. NON-GALACTIC STARS. $N = 509$.

$$\left(\frac{\overline{U_g}}{R}\right) = -2.170 \pm 0.218,$$

$$\left(\frac{\overline{V_g}}{R}\right) = -0.694 \pm 0.219,$$

$$\left(\frac{\overline{W_g}}{R}\right) = -0.847 \pm 0.300,$$

$$S\left(\frac{1}{R}\right) = 2.481 \pm 0.280,$$

$$\left(\frac{\bar{1}}{R}\right) = 0.588 \pm 0.067,$$

$$R = 1.72, \quad M = -1.17,$$

$$L = 17^{\circ}.7 \pm 5^{\circ}.8, \quad B = +20^{\circ}.4 \pm 5^{\circ}.4.$$

GROUP II. GALACTIC STARS. $N = 877$.

$$\left(\frac{\bar{U}_g}{R}\right) = -1.987 \pm 0.129,$$

$$\left(\frac{\bar{V}_g}{R}\right) = -0.640 \pm 0.129,$$

$$\left(\frac{\bar{W}_g}{R}\right) = -0.821 \pm 0.097,$$

$$S\left(\frac{\bar{1}}{R}\right) = 2.199 \pm 0.124,$$

$$\left(\frac{\bar{1}}{R}\right) = 0.528 \pm 0.049,$$

$$R = 1.89, \quad M = -1.89,$$

$$L = 18^{\circ}.8 \pm 3^{\circ}.4, \quad B = +21^{\circ}.9 \pm 3^{\circ}.2.$$

GROUP III. ALL STARS BRIGHTER THAN THE SIXTH MAGNITUDE. $N = 1386$.

$$\left(\frac{\bar{U}_g}{R}\right) = -2.045 \pm 0.127,$$

$$\left(\frac{\bar{V}_g}{R}\right) = -0.665 \pm 0.127,$$

$$\left(\frac{\bar{W}_g}{R}\right) = -0.826 \pm 0.117,$$

$$S\left(\frac{\bar{1}}{R}\right) = 2.304 \pm 0.126,$$

$$\left(\frac{\bar{1}}{R}\right) = 0.558 \pm 0.050,$$

$$R = 1.81, \quad M = -1.29,$$

$$L = 18^{\circ}.0 \pm 3^{\circ}.3, \quad B = +21^{\circ}.0 \pm 3^{\circ}.1.$$

If we compare these values of $\left(\frac{\bar{1}}{R}\right)$ with the corresponding values given in § 31, we find in each group systematically smaller values after the exclusion, which was to be expected. However, the differences between the two values for each group is smaller than the corresponding mean errors.

As this relatively strong exclusion has not materially altered the values of the mean value in question we have the less reason to exclude any star on account of large reduced proper motion.

33. For the three groups in question we have computed the mean reduced parallax for stars brighter than the sixth magnitude, or, with the notation used in the second chapter, the quantity

$$\left(\frac{\bar{1}}{R}\right)_{-\infty, m} \text{ for } m = 6.0.$$

We have there for this mean value got the expression (omitting the indices)

$$(17) \quad \left(\frac{\bar{1}}{R}\right) = e^{bM_0 + \frac{1}{2}b^2\sigma^2} \frac{A(m - b\sigma^2)}{A(m)}.$$

Here M_0 and σ are the mean value and the dispersion in the frequency function of the absolute magnitude, $A(m)$ the number of stars brighter than the apparent magnitude m , and $b = 0.2 \times \text{Mod.} = 0.46054$.

From (17) we shall now deduce the relation between M_0 and σ . The first thing to do is then to approximately represent $A(m)$ as a function of m , and with the help of this function form the ratio

$$\frac{A(m - b\sigma^2)}{A(m)}.$$

It will be worth observing that the mean value of $\frac{1}{R}$ is deduced from the parallax motion and that on account of this each square enters into the solution with a weight dependent on the distance from apex. To be strict we should have regard to this when forming $A(m)$ from the stars in the different squares, but as the number of stars varies very little from one square to another for each group and as the mean values found are approximately the same for the two groups, we may in this case neglect such a correction and suppose the mean value to be the same as if all the squares had contributed in equal degree to the determination.

We use the same division as before and get the following results.

GROUP I. NON-GALACTIC STARS.

TABLE IV.

Group I. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ computed	$O - C$
3.00	10	13	— 3
3.50	19	24	— 5
4.00	45	45	0
4.50	80	84	— 4
5.00	156	157	— 1
5.50	295	295	0
6.00	555	553	+ 2

In table IV column 2 are given the observed values of $A(m)$ for different m . When forming these values those stars mentioned in § 30 were excluded. The values of $A(m)$ given in column 3 are computed from

$$(38) \quad {}^{10}\log A(m) = 2.197 + 0.546(m - 5.00).$$

The fourth column contains the differences between the observed and the computed values of $A(m)$. From these differences we see that $A(m)$ is well represented by the simple equation (38).

From the table we notice that the number of stars brighter than the 6th magnitude is 555, whereas the corresponding number used for the determination of $\left(\frac{1}{R}\right)$ was 524. This is due to the wellknown fact that the catalogue of BOSS is not complete down to the magnitude 6.00 on the HARVARD-scale. For the stars treated here the catalogue in question may be considered as complete down to the HARVARD magnitude 5.90.

From equation (38) we at once find

$$\frac{A(m - b \sigma^2)}{A(m)} = 10^{-0.546 b \sigma^2} = e^{-2.78 b^2 \sigma^2}.$$

Introducing this value into (17) we get

$$\left(\frac{1}{R}\right) = e^{b M_0 - 2.23 b^2 \sigma^2}$$

from which follows

$$5 \cdot 10 \log \left(\frac{1}{R}\right) = M_0 - 2.23 b \sigma^2 = M_0 - 1.03 \sigma^2.$$

With the help of the value of $\left(\frac{1}{R}\right)$ given in (35*) this relation takes the form

$$(I) \quad M_0 = 1.03 \sigma^2 - 1.10.$$

We have thus for this group got M_0 as a linear function of σ^2 .

GROUP II. GALACTIC STARS.

TABLE V.

Group II. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ computed	$O - C$
3.00	19	17	+ 2
3.50	30	29	+ 1
4.00	49	52	— 3
4.50	106	102	+ 4
5.00	205	208	— 3
5.50	460	454	+ 6
6.00	1022	1045	— 23

$A(m)$ in the third column is calculated from

$$(39) \quad 10 \log A(m) = 2.319 + 0.650 (m - 5.00) + 0.050 (m - 5.00)^2.$$

There is a relatively great difference between the observed and the computed values for $m = 6.00$. But, as mentioned above, the catalogue of BOSS cannot be regarded as complete further than down to $m = 5.90$. For this magnitude we have $A(m) = 885$, and the corresponding value computed from (39) is 880, which is in good agreement.

With the help of (39) we get

$$\frac{A(m - b\sigma^2)}{A(m)} = e^{-b^2\sigma^2\{3.25 + 0.50(m - 5.00)\} + 0.25b^3\sigma^4}$$

and with regard to (17)

$$\left(\frac{1}{R}\right) = e^{bM_0 - b^2\sigma^2\{2.75 + 0.50(m - 5.00)\} + 0.25b^3\sigma^4}$$

We have here to put $m =$ the limiting magnitude $= 5.90$, and using the value (36*) of $\left(\frac{1}{R}\right)$ we then get the relation

$$(II) \quad M_0 = 1.47\sigma^2 - 0.058\sigma^4 - 1.22.$$

GROUP III. ALL STARS BRIGHTER THAN THE SIXTH MAGNITUDE.

TABLE VI.

Group III. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ computed	$O - C$
3.00	29	28	+ 1
3.50	49	51	- 2
4.00	94	95	- 1
4.50	186	182	+ 4
5.00	361	363	- 2
5.50	755	749	+ 6
6.00	1577	1600	- 23

The values given in column 3 are calculated from

$$(40) \quad {}^{10}\log A(m) = 2.560 + 0.614(m - 5.00) + 0.080(m - 5.00)^2,$$

which gives us, using the value (37*) of $\left(\frac{1}{R}\right)$

$$(III) \quad M_0 = 1.31\sigma^2 - 0.032\sigma^4 - 1.17.$$

In table VII the values of M_0 for different values of σ are calculated from the equations I, II and III.

TABLE VII.

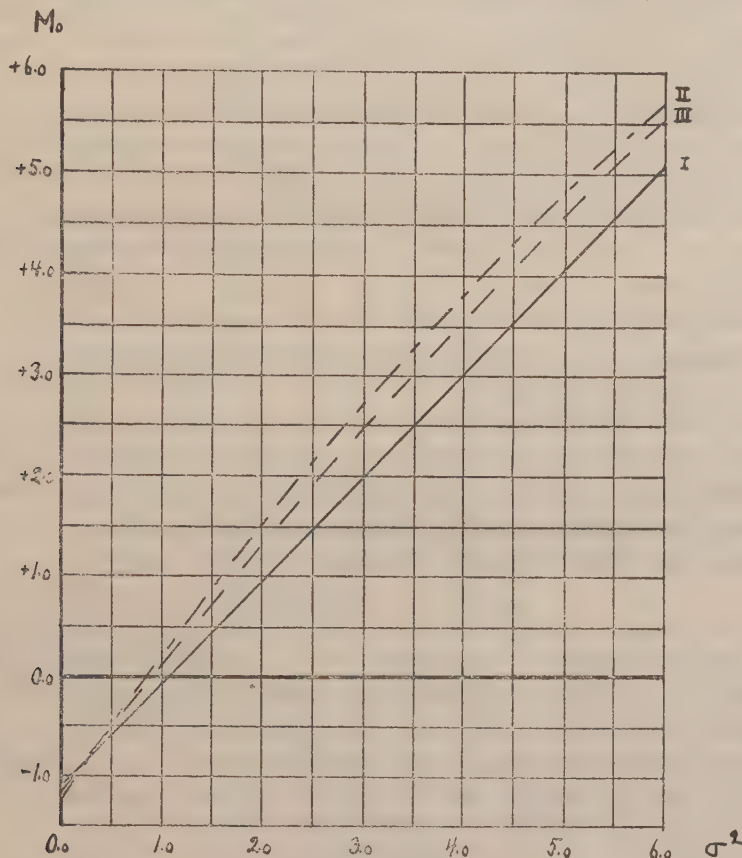
THE VALUES OF M_0 FOR DIFFERENT VALUES OF σ^2 .

σ^2	M_0^I	M_0^{II}	M_0^{III}	σ
0.0	- 1.10	- 1.22	- 1.17	0.0
0.5	- 0.58	- 0.50	- 0.51	0.7
1.0	- 0.07	+ 0.20	+ 0.11	1.0
2.0	+ 0.96	+ 1.51	+ 1.32	1.4
3.0	+ 1.99	+ 2.71	+ 2.47	1.7
4.0	+ 3.02	+ 3.81	+ 3.56	2.0
5.0	+ 4.05	+ 4.81	+ 4.58	2.2
6.0	+ 5.08	+ 5.69	+ 5.54	2.4

From the values given in this table a graphical representation of the equations I, II and III is constructed and given in fig. I.

FIG. I.

GRAPHICAL REPRESENTATION OF EQUATIONS I, II AND III.



As the curves given in fig. I show a very similar form, we may let the curve III represent both the galactic and the non galactic group. We find that the value of M_0 is closely dependent on the value of σ . If we should succeed in determining one of these constants we should at once get the corresponding value of the other. In chapter II a method was pointed out, which would eventually lead to a determination of σ . The BOSS stars, however, cannot give us but the above relations between M_0 and σ . It is true that in the first chapter we got for each subclass a greater value of R for stars brighter than 5^m_{00} than for those brighter than 6^m_{00} . But with regard to the relatively small number of stars in the former group and the form of the function $A(m)$ given above, we must ascribe this difference almost wholly to the influence of different errors.

B. THE STARS OF THE GREENWICH CATALOGUE FOR 1910.o. PART II.

34. *The Greenwich catalogue for 1910.o, Part II* contains the proper motions of the stars between $+24^\circ$ and $+32^\circ$ declination down to the ninth magnitude, as derived by F. W. DYSON and W. G. THACKERAY from a comparison of the Greenwich observations 1906 to 1915 with the *Astronomische Gesellschaft Catalogues* Leiden, Cambridge, Berlin B, and the observations of Lalande and Bessel. In consequence of the war, the publication of this catalogue has been delayed. But this delay has brought with it the great advantage that besides the positions and proper motions the spectra for most of the stars contained in the catalogue can also be published.

Thanks to the courtesy of Professor PICKERING and Miss CANNON these spectra (roughly for all stars brighter than $8^m.7$) are here given from the new HENRY DRAPER *Catalogue*, most of which is not yet published. In regard to magnitudes the HARVARD magnitude scale is used; for most of the stars the magnitudes have not been directly measured at HARVARD, but are taken from the *Bonner Durchmusterung* and reduced to the HARVARD magnitude scale with the help of table XIX in H. A. 72.

The catalogue is not yet published, but the material is discussed by F. W. DYSON and W. G. THACKERAY in several papers in the *Monthly Notices*. For the stars of the spectral types here treated these authors have computed the mean parallaxes between different limits of magnitude¹⁾. From these mean parallaxes they have derived the mean absolute magnitude from the relation

$$M = \bar{m} + 5 + 5 \cdot {}^{10}\log \bar{\pi},$$

where $\bar{m}$ is the mean apparent magnitude and $\bar{\pi}$ the mean parallax. The absolute magnitude used by them is the magnitude corresponding to the parallax $0''.1$. To reduce to our scale (the magnitude at a distance of one siriometer) we have to subtract $1^m.57$ from the absolute magnitudes given in the above-mentioned paper. After this reduction the values of M given by DYSON and THACKERAY are put together in table VIII.

TABLE VIII.
VALUES OF M FROM THE MEAN PARALLAX.

Limits of mag.	B_8-B_9				A_0			
	$\bar{m}$	M_1	M_2	N	$\bar{m}$	M_1	M_2	N
6.0—6.9	6.52	— 1.4	— 1.2	15	6.45	— 0.5	— 0.3	8
7.0—7.9	7.61	— 2.3	— 2.1	19	7.58	— 0.3	— 0.1	37
8.0—	8.46	— 1.9	— 1.7	69	8.67	— 0.8	— 0.6	160

M_1 are the values if the apex is assumed to be at $270^\circ + 35^\circ$, M_2 if at $270^\circ + 25^\circ$.

Though the resulting mean values are not strictly the mean values of M for the stars in question, which are given by the equation

$$M = \bar{m} + 5 + 5 \cdot {}^{10}\log \bar{\pi},$$

¹⁾ The Parallaxes of the B Stars which lie between the Limits of 4^h to 8^h R. A. and $+24^\circ$ to $+32^\circ$ Declination. M. N. 78, 651, 1918.

where $\overline{10 \log \pi}$ is the mean value of the logarithms of the parallax, yet these results support our view that the dispersion in absolute magnitude for the stars in question is small.

35. Realising that an examination of this material with respect to the reduced proper motions of the stars of types B8—A5 might perhaps lead to a value of σ , I applied to the Astronomer Royal Sir F. W. DYSON, who at my request had the great kindness to put at my disposal a proof of approximately half the *Greenwich Catalogue*, which was just in type, and for the rest I obtained an extract containing the positions, proper motions, and spectra of the stars of the types in question. As mentioned by DYSON and THACKERAY¹⁾ the spectra are given roughly for all stars brighter than 8^m.7.

I have chosen 8^m.5 as an upper limit in this treatise; an inspection of the first half of the catalogue, which was placed at my disposal in complete form, showed that the catalogue is practically complete up to this magnitude.

36. From the material of this catalogue we shall now compute the mean value of $\frac{1}{R}$. Owing to the smallness of the proper motions the trust-worthiness of our results depends to a very great extent upon the accuracy of the proper motions, and in particular any systematic error will be dangerous. To get the reduced proper motions we have to multiply the common proper motions by the photometric factor of reduction, $10^{0.2m}$. The magnitudes used must therefore agree with the photometric scale, otherwise our results will be misleading. Unfortunately the magnitudes used are not quite satisfactory, as they are based on the magnitudes in the *Bonner Durchmusterung*. But after the application of the corrections for the HARVARD magnitude-scale, we may assume that the systematic errors are practically eliminated.

From the proper motions and magnitudes the reduced proper motions

$$u' = 10^{0.2m} u, \quad v' = 10^{0.2m} v$$

$$(u = A \alpha \cos \delta, \quad v = A \delta)$$

were calculated for each star. To shorten the numerical calculations a method similar to that used in the case of the BOSS-stars was employed. The zone was divided into 48 squares of equal areas, the first between $\alpha = 0^h$ and $0^h.5$, the second between $0^h.5$ and $1^h.0$, and so on. The right ascension of the centre of each square is thus given from

$$\alpha_n = 0^h.25 + 0^h.50 (n - 1),$$

where n represents the order of the square. The declination of the centre of each square is put equal to $+28^\circ$.

In this case when we are only concerned with a zone in the equatorial system, it is not profitable to introduce the galactic system of coordinates. We calculate the direction cosines of the centre of each square referred to the common equatorial system K'' , also the quantities r_{ij} , which are defined in the first chapter.

¹⁾ Proper Motions of the Stars in Zone 24° — 32° N. Declination, in relation to Spectral Type. M. N. 79, 201, 1919.

Then the mean values, $\bar{u}'$ and $\bar{v}'$, of u' and v' are formed for each square. These mean values are given in table IX, which moreover contains the galactic latitude of the centre and the number of stars in each square.

TABLE IX.

STARS BRIGHTER THAN 8^m.
THE MEAN VALUES OF THE REDUCED PROPER MOTIONS FOR EACH SQUARE
OF THE GREENWICH CATALOGUE.

Square	Galactic Latitude	Number	$\bar{u}'$	$\bar{v}'$	Square	Galactic Latitude	Number	$\bar{u}'$	$\bar{v}'$
G_1	— 34°	17	+ 0".453	— 0".271	G_{25}	+ 84°	26	— 0".448	— 0".448
G_2	— 34°	13	+ 0".493	— 0".696	G_{26}	+ 89°	5	— 0".668	— 0".328
G_3	— 33°	14	+ 0".006	— 0".374	G_{27}	+ 82°	6	— 0".915	— 0".065
G_4	— 32°	18	+ 0".817	— 0".332	G_{28}	+ 76°	9	— 0".340	— 0".241
G_5	— 30°	15	+ 0".215	— 0".128	G_{29}	+ 69°	9	— 0".299	— 0".108
G_6	— 26°	29	+ 0".261	— 0".405	G_{30}	+ 62°	9	— 0".062	— 0".038
G_7	— 23°	30	+ 0".301	— 0".481	G_{31}	+ 56°	10	— 0".774	— 0".179
G_8	— 19°	49	+ 0".196	— 0".961	G_{32}	+ 49°	8	— 0".352	— 0".479
G_9	— 14°	23	+ 0".036	— 0".420	G_{33}	+ 43°	9	— 0".206	— 0".207
G_{10}	— 9°	29	— 0".161	— 0".478	G_{34}	+ 37°	11	— 0".013	— 0".441
G_{11}	— 4°	41	— 0".118	— 0".443	G_{35}	+ 30°	20	+ 0".058	+ 0".167
G_{12}	+ 2°	66	— 0".038	— 0".350	G_{36}	+ 24°	26	+ 0".088	— 0".321
G_{13}	+ 8°	59	+ 0".127	— 0".433	G_{37}	+ 18°	34	+ 0".137	+ 0".079
G_{14}	+ 14°	46	— 0".021	— 0".484	G_{38}	+ 12°	56	+ 0".090	+ 0".018
G_{15}	+ 20°	28	+ 0".233	— 0".585	G_{39}	+ 6°	66	+ 0".394	+ 0".104
G_{16}	+ 26°	20	— 0".161	— 0".779	G_{40}	+ 0°	104	+ 0".312	+ 0".054
G_{17}	+ 32°	23	— 0".379	— 0".865	G_{41}	— 5°	75	+ 0".326	+ 0".053
G_{18}	+ 39°	15	— 0".144	— 0".667	G_{42}	— 11°	56	+ 0".256	+ 0".045
G_{19}	+ 45°	12	+ 0".020	— 0".771	G_{43}	— 16°	31	+ 0".414	+ 0".168
G_{20}	+ 52°	10	— 0".850	— 0".377	G_{44}	— 20°	33	— 0".072	— 0".139
G_{21}	+ 58°	5	— 0".418	— 0".028	G_{45}	— 24°	28	+ 0".376	— 0".220
G_{22}	+ 65°	8	— 0".961	— 0".476	G_{46}	— 28°	21	+ 0".066	— 0".296
G_{23}	+ 71°	8	— 0".100	— 0".280	G_{47}	— 30°	24	+ 0".058	— 0".210
G_{24}	+ 78°	10	+ 0".071	+ 0".026	G_{48}	— 32°	12	+ 0".188	— 0".034

With the help of the calculated quantities we get two equations for each square, which are analogous to equations (34*) and are of the form

$$\bar{u}' = \gamma_{11} \left(\frac{\bar{U}''}{R} \right) + \gamma_{21} \left(\frac{\bar{V}''}{R} \right),$$

$$\bar{v}' = \gamma_{12} \left(\frac{\bar{U}''}{R} \right) + \gamma_{22} \left(\frac{\bar{V}''}{R} \right) + \gamma_{32} \left(\frac{\bar{W}''}{R} \right).$$

From these the corresponding normal equations are formed. When forming the normal equations I have proceeded in a manner differing somewhat from that hitherto used. As the number of stars in the different squares varies very greatly—from 5 in the neighbourhood of the north galactic pole to 104 in the galactic plane—I have, in forming the normal equations, given each square a weight proportional to the number of stars in the square. Solving the normal equations we get the values of

$$\left(\frac{\bar{U}''}{R} \right), \left(\frac{\bar{V}''}{R} \right), \left(\frac{\bar{W}''}{R} \right).$$

If we suppose the velocity of the sun to be the same as before—i. e. that the system of stars in question is at rest with respect to the system of the brighter stars—we get from these values the mean reduced parallax for the stars in question. With the help of (6*) we get the position of the apex. When forming $\bar{u}'$ and $\bar{v}'$ I have not excluded any star on account of large reduced motion. In regard to double stars I have proceeded in the same manner as in the case of the BOSS stars. The material used consists of 1276 stars of the spectral types B8—A5 brighter than 8^m.5.

37. From this material we get the following results:—

$$\begin{aligned}\left(\frac{\bar{U}''}{R}\right) &= +0.100 \pm 0.228, \\ \left(\frac{\bar{V}''}{R}\right) &= +2.085 \pm 0.275, \\ \left(\frac{\bar{W}''}{R}\right) &= -1.412 \pm 0.219, \\ S\left(\frac{\bar{1}}{R}\right) &= 2.520 \pm 0.259,\end{aligned}$$

with the value $S = 4.165 \pm 0.805$ we get

$$\begin{aligned}\left(\frac{\bar{1}}{R}\right) &= 0.605 \pm 0.076, \\ R &= 1.65, \quad M = -1.09, \\ A &= 267^{\circ}.3 \pm 7^{\circ}.1, \quad D = +34^{\circ}.1 \pm 5^{\circ}.9, \\ L &= 28^{\circ}.5, \quad B = +25^{\circ}.1.\end{aligned}$$

If we compare the above value of $\left(\frac{\bar{1}}{R}\right)$ with the corresponding value for all the BOSS stars, we find the value for the *Greenwich* stars to be a little larger. For the BOSS stars we had (37*)

$$\left(\frac{\bar{1}}{R}\right) = 0.584 \pm 0.051.$$

In order to examine whether the galactic and non-galactic stars give us different values or not, I have divided the material into two groups, the first containing all squares the centres of which have a galactic latitude $> +30^{\circ}$ or $< -30^{\circ}$, the second containing those for which the centres has a galactic latitude between $+30^{\circ}$ and -30° . We get the following values:—

Group a. Gal. latitude $> +30^{\circ}$ or $< -30^{\circ}$. 23 squares. $N = 311$.

$$\begin{aligned}\left(\frac{\bar{U}''}{R}\right) &= -0.165 \pm 0.470, \\ \left(\frac{\bar{V}''}{R}\right) &= +1.780 \pm 0.370, \\ \left(\frac{\bar{W}''}{R}\right) &= -1.774 \pm 0.366,\end{aligned}$$

$$S\left(\frac{\bar{1}}{R}\right) = 2.518 \pm 0.368, \quad \left(\frac{\bar{1}}{R}\right) = 0.804 \pm 0.099,$$

$$R = 1.66, \quad M = -1.10,$$

$$A = 275^{\circ}.8 \pm 11^{\circ}.8, \quad D = +44^{\circ}.8 \pm 8^{\circ}.4,$$

$$L = 40^{\circ}.0, \quad B = +22^{\circ}.5.$$

Group b. Gal. latitude between $+30^{\circ}$ and -30° . 23 squares. $N = 965$.

$$\left(\frac{\bar{U}''}{R}\right) = +0.171 \pm 0.282,$$

$$\left(\frac{\bar{V}''}{R}\right) = +2.210 \pm 0.392,$$

$$\left(\frac{\bar{W}''}{R}\right) = -1.250 \pm 0.284,$$

$$S\left(\frac{\bar{1}}{R}\right) = 2.545 \pm 0.368, \quad \left(\frac{\bar{1}}{R}\right) = 0.611 \pm 0.099,$$

$$R = 1.64, \quad M = -1.07,$$

$$A = 265^{\circ}.6 \pm 9^{\circ}.6, \quad D = +29^{\circ}.4 \pm 8^{\circ}.8,$$

$$L = 21^{\circ}.5, \quad B = +25^{\circ}.0.$$

If we compare the results for the two groups in question we find the same value for $\left(\frac{\bar{1}}{R}\right)$, but a considerable variation in the coordinates of the apex. This variation, however, is not too great to be attributable to accidental errors. We have indeed assumed that the system formed by each group is at rest with respect to the system formed by the brighter stars. This assumption is of course never exactly fulfilled, especially if the material considered is small. In such a case local irregularities may cause rather large deviations. If we calculate the galactic coordinates of the apex we find that the deviations are almost wholly in galactic longitude. The declination found for all the Greenwich stars agree with those found by DYSON and THACKERAY¹⁾ from the proper motions. The latter values are given in table X.

TABLE X.
THE COORDINATES OF THE APEX DERIVED FROM THE PROPER MOTIONS.

Spectral type	Number of stars	A	D
B 8—B 9	289	283°	$+33^{\circ}$
A 0	732	275°	$+34^{\circ}$
A 2	482	282°	$+34^{\circ}$
A 3	174	257°	$+41^{\circ}$
A 5	165	270°	$+29^{\circ}$

On the other hand, the value of declination of the apex found from these stars deviates from the value obtained for the BOSS stars, from which we found

$$A = 265^{\circ}.8, \quad D = +25^{\circ}.5.$$

¹⁾ M. N. 79, 206.

From his catalogue L. BOSS¹⁾ has obtained as the coordinates of the apex for 1647 stars of the spectral types B 8—A 4

$$A = 270^{\circ}_0, \quad D = +28^{\circ}_8$$

and CHARLIER and GYLLENBERG²⁾ from 1281 A-stars brighter than the sixth magnitude

$$A = 266^{\circ}_9, \quad D = +26^{\circ}_2,$$

values which agree with those found by me.

The difference between the values of the coordinates of the apex found from the *Greenwich* stars and the BOSS stars also lies almost wholly in galactic longitude.

38. The established difference in the coordinates of the apex, however, does not seem to have any perceptible influence on the values of the mean reduced parallax.

From the two different groups of the *Greenwich* stars we got the same value for $\left(\frac{\bar{1}}{R}\right)$, but somewhat diverging values for the coordinates of the apex. I have now recalculated $\left(\frac{\bar{1}}{R}\right)$

for the two groups above under the supposition of a given apex. As values of the coordinates of the apex I assume $A = 270^{\circ}$, $D = +30^{\circ}$ partly because this value is situated approximately half-way between those found from the brighter and fainter stars, partly because the above value is the one most usually assumed for the apex. We divide the reduced proper motions into two components, ν' towards the antapex and τ' at right angles thereto.

Forming the mean value of the component ν' for each square, we get for the determination of the mean reduced parallax, if λ is the angle between the centre of a square and the apex, an equation of the form

$$(41) \quad \sin \lambda \cdot S\left(\frac{\bar{1}}{R}\right) = \nu'$$

for each square.

Equations (41) are solved by the method of least squares. In the formation of the resulting normal equations a weight equivalent to the number of stars was given to each square.

The following results were obtained:—

Group a. $N = 311$.

$$\begin{aligned} S\left(\frac{\bar{1}}{R}\right) &= 2.555 \pm 0.306, & \left(\frac{\bar{1}}{R}\right) &= 0.613 \pm 0.086, \\ R &= 1.68, & M &= -1.06. \end{aligned}$$

Group b. $N = 965$.

$$\begin{aligned} S\left(\frac{\bar{1}}{R}\right) &= 2.589 \pm 0.288, & \left(\frac{\bar{1}}{R}\right) &= 0.621 \pm 0.086, \\ R &= 1.61, & M &= -1.08. \end{aligned}$$

¹⁾ Precession and Solar Motion. Third paper. A. J. 26, 187, 1911.

²⁾ Die Bewegung der Sterne verschiedener Spektraltypen. L. M. I, 92, 1919.

Group c. All stars. $N = 1276$.

$$(42) \quad S\left(\frac{\bar{1}}{R}\right) = 2.577 \pm 0.158, \quad \left(\frac{\bar{1}}{R}\right) = 0.618 \pm 0.063.$$

$$R = 1.62; \quad M = -1.04.$$

We thus get the same value for the mean reduced parallax in the different groups. This value agrees with those obtained by the usual method. It is true that the value obtained here is somewhat larger, but the difference is inconsiderable compared with the mean errors.

39. From the mean reduced parallax we shall now form the relation between M_0 and σ with the aid of equation (17). It was pointed out in connection with the discussion of the BOSS stars that in the solution of the equations the squares have got different weights, dependent on their position with respect to the apex, and that we have to pay attention to this fact when forming the ratio $\frac{A(m - b \sigma^2)}{A(m)}$ in (17) for the group in question. For the BOSS stars, which form a homogenous material, we have neglected to introduce any weights when forming $A(m)$. In the case of the *Greenwich* stars we must however make allowance for the different weights, then, as has been pointed out above, the number of stars per square varies considerably. The "reduced" value of $A(m)$ is calculated in the following way, the value of the apex being for simplicity's sake assumed as given. We have to determine $\left(\frac{\bar{1}}{R}\right)$ for each square by means of an equation of the form (41)

$$\sin \lambda_n \cdot S\left(\frac{\bar{1}}{R}\right)_n = \bar{\nu}_n'.$$

When solving the mean value for all the squares we have used the method of least squares and have given to each square a weight proportional to the number of stars ($= N$).

The resulting mean value is also given from

$$(43) \quad \left(\frac{\bar{1}}{R}\right) = \frac{\sum N \sin^2 \lambda_n \left(\frac{\bar{1}}{R}\right)_n}{\sum N \sin^2 \lambda_n}.$$

But, according to (17) we have for each square

$$(44) \quad \left(\frac{\bar{1}}{R}\right)_n = e^{b M_0 + \frac{1}{2} b^2 \sigma^2} \frac{A_n(m - b \sigma^2)}{A_n(m)},$$

where $A_n(m) = N$ and M_0 and σ are considered to have the same values for all the squares.

If we introduce the value (44) for $\left(\frac{\bar{1}}{R}\right)_n$ into (43), this equation takes the form

$$(44^*) \quad \left(\frac{\bar{1}}{R}\right) = e^{b M_0 + \frac{1}{2} b^2 \sigma^2} \frac{\sum \sin^2 \lambda_n A_n(m - b \sigma^2)}{\sum \sin^2 \lambda_n A_n(m)}.$$

When forming the function $A(m)$ for the combined squares, we have thus to multiply the values of $A(m)$ for each square by $\sin^2 \lambda_n$, where λ_n is the angle between the square and the apex, and to sum these products.

TABLE XI.
THE GREENWICH STARS. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ reduced	$A(m)$ computed	$R - C$
6.05	120	136	136	0
6.55	188	211	213	— 2
7.05	328	352	333	+19
7.55	498	518	521	— 3
8.05	818	822	817	+ 5
8.55	1276	1276	1276	0

The result of this reduction for the Greenwich stars is given in table XI. Column 2 gives the observed values of $A(m)$ between 6^m.0 and 8^m.5, column 3 contains the "reduced" value. In order to obtain a direct comparison with the observed values, the "reduced" values have been multiplied by a constant factor; thus it has been chosen that the two values of $A(m)$ shall agree for $m = 8.5$. The values of $A(m)$ in the fourth column are computed from the formula

$$(45) \quad {}^{10}\log A(m) = 3.106 + 0.389 (m - 8.55).$$

In the last column the differences between the "reduced" and computed values of $A(m)$ are given.

From (45) we get

$$A(m) = C \cdot e^{1.945 b m}$$

and thus

$$(45^*) \quad \frac{A(m - b \sigma^2)}{A(m)} = e^{-1.945 b^2 \sigma^2}.$$

With the help of equation (17) we get

$$(46) \quad \left(\frac{1}{R} \right) = e^{b M_0 + \frac{1}{2} b^2 \sigma^2 - 1.945 b^2 \sigma^2} = e^{b M_0 - 1.445 b^2 \sigma^2}.$$

Using the value for $\left(\frac{1}{R} \right)$ obtained above assuming a given apex, viz.

$$\left(\frac{1}{R} \right) = 0.618, \text{ we get from (46)}$$

$$(IV) \quad M_0 = 1.445 b \sigma^2 - 1.04 = 0.665 \sigma^2 - 1.04.$$

From this relation the values in table XII are calculated.

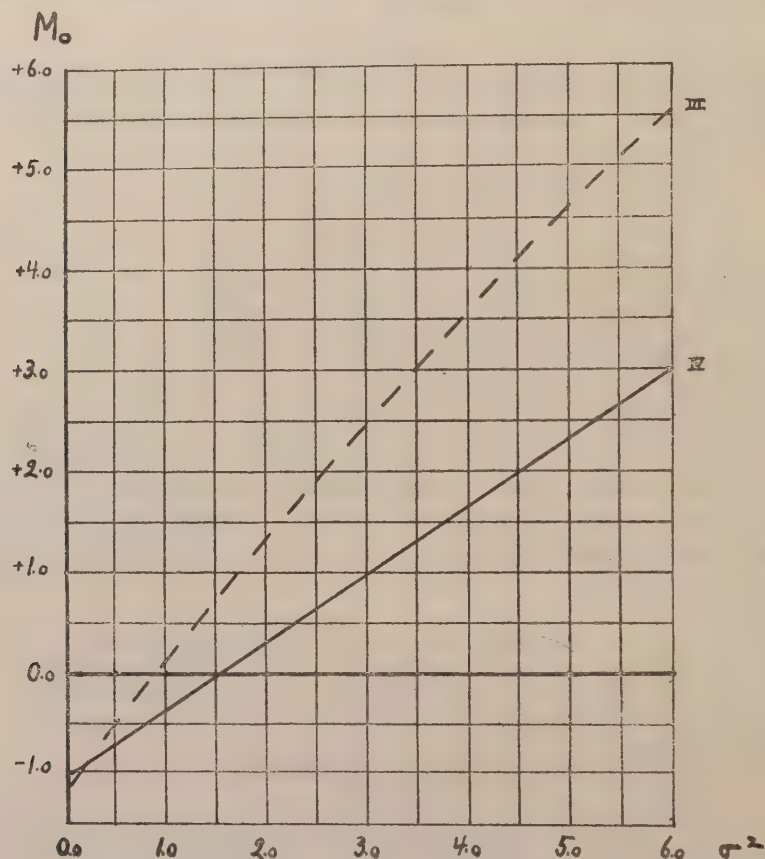
By means of this table equation IV is graphically represented in fig. II. For the sake of comparison the corresponding curve obtained from the BOSS stars curve III in fig. I) is also given in the same figure.

TABLE XII.
VALUES OF M_0 FOR
DIFFERENT VALUES OF σ .

σ^2	M_0
0.0	— 1.04
0.5	— 0.71
1.0	— 0.38
2.0	+ 0.29
3.0	+ 0.96
4.0	+ 1.62
5.0	+ 2.28
6.0	+ 2.95

FIG. II.

GRAPHICAL REPRESENTATION OF THE EQUATIONS III AND IV.



40. The intersection between the two curves would give us the required values of M_0 and σ . We should then have

$$M_0 = -0.91, \quad \sigma = 0.44, \quad (\sigma^2 = 0.196).$$

This value of σ is undoubtedly too small.

The mean errors in the determination of $\left(\frac{1}{R}\right)$ are considerable, and the form used for $A(m)$ is only an approximation. We must therefore wait for more copious material before we can get a reliable result. The only result of the present discussion is the knowledge that the dispersion cannot be great. We may try to fix an upper limit for σ in the following way. We use the relation (21), which can be written in the form

$$\frac{A(m_1 - b\sigma^2)}{A(m_1)} : \frac{A(m_2 - b\sigma^2)}{A(m_2)} = \frac{\left(\frac{1}{R}\right)_{-\infty, m_1}}{\left(\frac{1}{R}\right)_{-\infty, m_2}}.$$

But supposing the velocity of the sun to be the same we have

$$\frac{\left(\frac{1}{R}\right)_{-\infty, m_1}}{\left(\frac{1}{R}\right)_{-\infty, m_2}} = \frac{S\left(\frac{1}{R}\right)_{-\infty, m_1}}{S\left(\frac{1}{R}\right)_{-\infty, m_2}},$$

and this ratio is then independent of the value of the velocity of the sun.

Thus we may write

$$(47) \quad \frac{A(m_1 - b \sigma^2)}{A(m_1)} : \frac{A(m_2 - b \sigma^2)}{A(m_2)} = \frac{S\left(\frac{1}{R}\right)_{-\infty, m_1}}{S\left(\frac{1}{R}\right)_{-\infty, m_2}} = k.$$

Now if the index 1 corresponds to the BOSS stars, and the index 2 to the *Greenwich* stars, we get from (40)

$$\frac{A(m_1 - b \sigma^2)}{A(m_1)} = e^{-3.34 b^2 \sigma^2 + 0.150 b^3 \sigma^4}$$

and from (45*)

$$\frac{A(m_2 - b \sigma^2)}{A(m_2)} = e^{-1.94 b^2 \sigma^2}.$$

Further is

$$S\left(\frac{1}{R}\right)_{-\infty, m_1} = 2.433 \pm 0.112$$

and

$$S\left(\frac{1}{R}\right)_{-\infty, m_2} = 2.577 \pm 0.186,$$

from which it follows that

$$\frac{S\left(\frac{1}{R}\right)_{-\infty, m_1}}{S\left(\frac{1}{R}\right)_{-\infty, m_2}} = 0.944 \pm 0.080.$$

From (47) we thus get

$$(48) \quad e^{-1.40 b^2 \sigma^2 + 0.150 b^3 \sigma^4} = k = 0.944 \pm 0.080.$$

For a given value of k we get from (48) two values for σ . But as this equation is founded on an approximation of $A(m)$, valid only for a limited range of m , we can only use the lower value of σ . The other value is moreover so great as to be out of the question. Taking into consideration the errors from different sources introduced into (48) we may say that the true value of k will lie less than four times the mean error from the value obtained. As further the value of k cannot exceed unity we may assume

$$1 > k > 0.624.$$

The value $k = 1$ gives us $\sigma = 0$, the value $k = 0.624$ $\sigma = 1^m_{32}$. The corresponding value of M_0 is found from fig. 2. As the BOSS stars afford the most trustworthy

material both in regard to proper motions and to magnitudes. I use equation III when determining M_0 . We get for

$$\sigma = 0, \quad M_0 = -1.17,$$

and for

$$\sigma = 1.82, \quad M_0 = +1.01.$$

41. From the material used in sections A and B we can only conclude in regard to the constants M_0 and σ that:—

1. The dispersion will probably not exceed 1^ms, the material in question indicates a considerably lower value.

2. The mean value of the absolute magnitude is closely dependent on the value of the dispersion. As soon as the value of the latter constant is determined, we get the corresponding value of M_0 from equation III.

C. THE VALUES OF M_0 AND σ FROM THE MEAN PARALLAX.

42. As was pointed out in the second chapter, KAPTEYN¹⁾ and his collaborators have recently determined the secular parallax of stars of different magnitude, galactic latitude, and spectrum, down to the ninth magnitude, from material brought together from all the sources at their disposal. The result of this investigation is that down to the ninth magnitude the secular parallax of the stars of each spectral type may be written in the form of

$$10 \log \left(\frac{\bar{S}}{r} \right) = a - a_2 m,$$

where S is the velocity of the sun and r the distance. The coefficient a_2 is found to be a constant for all spectral types, the coefficient a varies with the spectral type and—though inconsiderably—with the galactic latitude. For the A-stars the relation has the form

$$(49) \quad 10 \log \left(\frac{\bar{S}}{r} \right) = 9.577 - 0.0186 \cos 2b - 0.1697 m,$$

where b is the galactic latitude.

By the aid of the known value of the velocity of the sun we may transform the secular parallax into the annual. Using our value of the sun's velocity

$$S = 4.167 \text{ sir./st.},$$

we have to multiply the secular parallax by

$$\frac{1}{S} = 0.240$$

in order to get the annual parallax. Equation (49) then takes the form of

$$(50) \quad 10 \log \bar{\pi}_m = 8.957 - 0.0186 \cos 2b - 0.1697 m.$$

¹⁾ G. P. 29.

In the second chapter we have shown that, if we assume the frequency functions both of the apparent and absolute magnitudes to be normal curves of type A, we get $\bar{\pi}_m$ under the form (27), viz. the form found by KAPTEYN.

If we then assume

$$a(m) = \frac{N}{\alpha\sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\alpha^2}}$$

and

$$\varphi(M) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0)^2}{2\sigma^2}},$$

we get according to (30**), putting $1 - \frac{\sigma^2}{\alpha^2} = \lambda_1$,

$$(51) \quad \begin{cases} 5a_1 = M_0 - 3.48 - m_0(1 - \lambda_1) + \frac{1}{2}b\sigma^2\lambda_1, \\ 5a_2 = \lambda_1, \end{cases}$$

where according to (50)

$$\begin{aligned} a_1 &= -1.048 - 0.0186 \cos 2b, \\ a_2 &= 0.1697, \end{aligned}$$

from which we at once get

$$\lambda_1 = 0.848.$$

Hence follows

$$(51^*) \quad \frac{\sigma^2}{\alpha^2} = 1 - \lambda_1 = 0.152.$$

As the coefficient of m in formula (50) was found to be the same for all spectral types, the value of $\frac{\sigma^2}{\alpha^2}$ would be a constant for all spectral types. If we could succeed in determining the value of α we should get from (51*) the corresponding value of σ . There are, however, great difficulties in fixing this value. Even if the stars down to the 17th magnitude are considered, we have still not reached the top of the curve, taking all spectral types together. For the A-stars alone the difficulty is somewhat less, thanks to the relatively great intrinsic brightness of these stars.

43. As has been pointed out by DYSON and THACKERAY¹⁾ the A-stars in the *Greenwich Catalogue* show a very great diminution in number as the distance from the Galaxy increases. These writers find the following values for the stars of the types B 8, B 9 and A 0:

Mean gal. lat.	Number of stars
+ 80°	19
+ 10°	279
+ 3°	340
— 33°	61.

If we confine ourselves to stars brighter than 8^m, for which the catalogue in question may be considered complete, we get for the stars of types B 8—A 5 the values given in the following table.

¹⁾ M. N. 79, page 201.

TABLE XIII.
NUMBER OF STARS FOR DIFFERENT
GALACTIC LATITUDES.

Gal. lat.	Number per square	Number per $\square^\circ$
+ 90° — + 60°	10.0	0.19
+ 60° — + 40°	9.0	0.17
+ 40° — + 20°	19.2	0.36
+ 20° — 0°	57.4	1.08
0° — - 20°	49.0	0.93
- 20° — - 40°	21.2	0.40

($\square^\circ$ = square—degree.)

given apparent magnitude, and thus we get a greater part of the curve from stars around the galactic poles than from stars in the neighbourhood of the galactic plane.

44. The problem to be solved is the following. We know the number of stars of each magnitude up to a certain limit. From these numbers we have to determine the constants in the function $a(m)$ under the assumption that

$$(52) \quad a(m) = \frac{N}{\alpha \sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\alpha^2}}.$$

This problem was solved by CHARLIER¹⁾ in the following way.

The number of stars brighter than a given apparent magnitude m is given from

$$A(m) = \frac{N}{\alpha \sqrt{2\pi}} \int_{-\infty}^m dm e^{-\frac{(m-m_0)^2}{2\alpha^2}}.$$

Put

$$m - m_0 = \alpha \cdot x, \therefore dm = \alpha \cdot dx,$$

and we have

$$A(m) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m-m_0}{\alpha}} dx e^{-\frac{x^2}{2}}.$$

We now introduce the probability integral $P(x)$, defined through the formula

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{+x} dx e^{-\frac{x^2}{2}} = \frac{2}{\sqrt{2\pi}} \int_0^x dx e^{-\frac{x^2}{2}},$$

by means of which we get

$$(52^*) \quad A(m) = \frac{N}{2} \left\{ 1 - P\left(\frac{m_0 - m}{\alpha}\right) \right\}$$

or, putting

$$\frac{m_0}{\alpha} = x, \quad \frac{1}{\alpha} = y,$$

(52**)

$$P(x - my) = 1 - \frac{2 A(m)}{N}.$$

¹⁾ L. M. II, 8.

Here too we find a strong concentration towards the Galaxy, though this is not so pronounced as when the subclasses B 8, B 9 and A 0 only are considered. The diminution in number with increasing galactic latitude supports the view that the A-stars form an oblate system, the extension of which is considerably smaller towards the galactic poles than in the galactic plane. When examining the form of the function $a(m)$, we shall therefore get the most reliable results for the stars around the galactic poles. Our material reaches only up to a

The inversion of the probability integral is by CHARLIER called the error-function. This function is tabulated in L. M. II. 8. If

$$u = P(x),$$

we thus have

$$x = Err(u).$$

From (52**) we then get

$$x - my = Err\left(1 - \frac{2 A(m)}{N}\right).$$

For the determining of x , y and N we ought to have three such equations,

$$(53) \quad \begin{cases} x - m_1 y = Err\left(1 - \frac{2 A(m_1)}{N}\right), \\ x - m_2 y = Err\left(1 - \frac{2 A(m_2)}{N}\right), \\ x - m_3 y = Err\left(1 - \frac{2 A(m_3)}{N}\right). \end{cases}$$

The value of N is then obtained from the equation

$$(53^*) \quad 0 = \begin{vmatrix} 1, m_1, Err\left(1 - \frac{2 A(m_1)}{N}\right) \\ 1, m_2, Err\left(1 - \frac{2 A(m_2)}{N}\right) \\ 1, m_3, Err\left(1 - \frac{2 A(m_3)}{N}\right) \end{vmatrix} = (m_3 - m_2) Err\left(1 - \frac{2 A(m_1)}{N}\right) + \\ + (m_1 - m_3) Err\left(1 - \frac{2 A(m_2)}{N}\right) + \\ + (m_2 - m_1) Err\left(1 - \frac{2 A(m_3)}{N}\right).$$

To solve this equation we may give to N a series of increasing values until the right member of (53*) changes its sign. When N is thus determined we get the other two unknowns from any two of the equations (53).

45. By means of this method I have tried to determine the constants of (52) from the stars in the *Greenwich Catalogue*, and have for this purpose used those squares which are situated more than 30° from the galactic equator. The values of $A(m)$ for different magnitudes are given in the second column of table XIV.

TABLE XIV.

Group a. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ computed	$O - C$
3.05	2	1	+ 1
4.05	5	5	0
5.05	10	16	- 6
5.55	26	27	- 1
6.05	49	46	+ 3
6.55	72	72	0
7.05	114	110	+ 4
7.55	162	162	0
8.05	221	229	- 8
8.55	311	311	0
9.05	406	408	- 2
9.55	(461)	517	-

Putting $m_1 = 6.55$, $m_2 = 7.55$, $m_3 = 8.55$, we get

$$\begin{aligned} N &= 1400 \text{ (per } \square^\circ = 1.058), \\ y &= 0.4824 \cdot \therefore \alpha = 2.81, \\ x &= 4.4625 \cdot \therefore m_0 = 10.32, \end{aligned}$$

we get for $a(m)$ the expression

$$(54) \quad a(m) = \frac{1400}{2.81 \sqrt{2\pi}} e^{-\frac{(m-10.32)^2}{2 \cdot (2.81)^2}}.$$

Using this form of $a(m)$ the values of $A(m)$ given in the third column are calculated by the aid of (52*). From the fourth column, where the differences between the observed and the computed values are given, it will be seen that formula (54) represents the observations very well. From the above table we see that, if (54) is adequate to the real distribution of m , the catalogue used will be complete down to the ninth magnitude for the squares considered.

46. Fortunately we are able to perform the same computation for the stars around the south galactic pole. The great undertaking of the HARVARD *Observatory*—the classification of the spectra of the stars down to the ninth magnitude—is now completed, and the first part of the great HENRY DRAPER *Catalogue*, containing the stars between $0^h 0^m$ and $4^h 0^m$ right ascension, is now published¹⁾ (H. A. 91). As the south galactic pole ($\alpha = 10^\circ$, $\delta = -28^\circ$) is included in this part we may study the distribution of the stars about this pole from the data of this catalogue. I have made a card-catalogue of the stars of the subclasses B 8 to A 5 in the Draper Catalogue. With the help of a map for converting equatorial coordinates into galactic ones I have divided the material into five zones between different galactic latitudes. The limits of latitude for the different zones are approximately I. -90° — -60° ; II. -60° — -40° ; III. -40° — -20° ; IV. -20° — 0° ; V. 0° — $+20^\circ$. The number of stars of each sub-class brighter than 8^m_5 (on the photometric scale) is given in table XV. The numbers are given per $\square^\circ$, in the last column the areas in $\square^\circ$ of the different zones are given. The zones of course do not contain all the stars between the above limits, but only those whose right ascension lies between $0^h 0^m$ and $4^h 0^m$.

TABLE XV.

NUMBER OF STARS BRIGHTER THAN 8^m_5 BETWEEN DIFFERENT LIMITS OF GALACTIC LATITUDE.

Zone	Limits of gal. lat.	$A(m)$ per $\square^\circ$							Area in $\square^\circ$
		B 8	B 9	A 0	A 2	A 3	A 5	B 8—A 5	
I	-90° — -60°	0.005	0.008	0.036	0.086	0.026	0.026	0.137	1837
II	-60° — -40°	0.009	0.017	0.066	0.058	0.040	0.039	0.229	2136
III	-40° — -20°	0.027	0.042	0.155	0.094	0.048	0.052	0.418	1302
IV	-20° — 0°	0.076	0.187	0.418	0.195	0.083	0.058	1.017	899
V	0° — $+20^\circ$	0.098	0.128	0.326	0.148	0.087	0.050	0.832	481

¹⁾ Since this sentence was written, two more parts of the catalogue have appeared.

From the table we see that the increase in the number of stars as we approach the galactic equator is most pronounced for the sub-classes B 8 and B 9 and becomes then successively weaker. Whether this fact is due to a real difference in the distribution in space of the stars of the different subclasses or is only a consequence of a difference in their absolute brightness cannot be settled as yet. The values found in this investigation for the mean absolute and apparent magnitudes indicate a considerably greater extension towards the galactic poles of the A-stars than that found by CHARLIER¹⁾ for the B-stars. But we must wait for more abundant material before we can draw any definite conclusions.

47. From the stars in zone I we get the values of $A(m)$ given in column 2 of table XVI. Putting $m_1 = 6.05$, $m_2 = 7.55$, $m_3 = 9.05$ we get

$$N = 940 \text{ (per } \square^\circ = 0.512)$$

$$y = 0.4929, \quad \therefore \alpha = 2.03,$$

$$x = 4.8094, \quad \therefore m_0 = 9.76$$

and then

$$(55) \quad a(m) = \frac{940}{2.03\sqrt{2\pi}} e^{-\frac{(m-9.76)^2}{2 \cdot (2.03)^2}}.$$

From this relation the values of $A(m)$ in column 3 are computed. In the fourth column the differences between the observed and computed values are given.

TABLE XVI.
Zone I. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ computed	$O - C$
3.05	0	0	0
4.05	2	2	0
5.05	10	10	0
5.55	22	18	+ 4
6.05	32	32	0
6.55	52	53	- 1
7.05	84	85	- 1
7.55	130	130	0
8.05	180	188	- 8
8.55	253	259	- 6
9.05	342	341	+ 1
9.55	458	431	+27

We see that the formula (55) fits the observed values well up to the ninth magnitude, for 9^m₅₅ the difference is somewhat great. But as the fainter magnitudes are not very trustworthy, we cannot take the difference too seriously.

From zone II we get, putting $m_1 = 6.05$, $m_2 = 7.55$, $m_3 = 8.55$.

$$N = 3000 \text{ (per } \square^\circ = 1.404),$$

$$y = 0.4864, \quad \therefore \alpha = 2.29,$$

$$x = 4.7083, \quad \therefore m_0 = 10.78.$$

$$(56) \quad a(m) = \frac{3000}{2.29\sqrt{2\pi}} e^{-\frac{(m-10.78)^2}{2 \cdot (2.29)^2}}.$$

¹⁾ L. M. II, 14.

In table XVII the data for this zone are given. The values of $A(m)$ calculated with the help of (56) are found in column 3; from column 4, where the differences between the observed and calculated values are given, we see that the agreement is not so good as for the previous materials.

TABLE XVII.

Zone II. NUMBER OF STARS BRIGHTER THAN m .

m	$A(m)$ observed	$A(m)$ computed	$O - C$
3.05	0	1	- 1
4.05	2	5	- 3
5.05	20	19	+ 1
5.55	37	34	+ 3
6.05	58	58	0
6.55	87	97	-10
7.05	170	155	+15
7.55	238	238	0
8.05	332	350	-18
8.55	497	497	0
9.05	706	675	+31

48. As the computations of the constants of $a(m)$ are based on observations covering only a relatively small part of the curve, the values found are a little unsafe. The values of α are, however, fairly concordant, and we may use the mean value of those found above. We then get

$$\alpha = 2.21.$$

From the relation (51*)

$$\frac{\sigma^2}{\alpha^2} = 0.152,$$

we get

$$\sigma = 0.390\alpha$$

and hence it follows, with the above value of α , that

$$(57) \quad \sigma = 0^{\text{m}}_{.86}.$$

When σ is known we may from (51) calculate the value of M_0 . We then need the value of m_0 . As an approximate value we may use the mean of the values found here, *viz.*

$$m_0 = 10^{\text{m}}_{.3}.$$

With this value of m_0 we get, putting $b = 60^\circ$,

$$(57^*) \quad M_0 = -0^{\text{m}}_{.32}.$$

From the value (57) of σ we may also get the corresponding value of M_0 from the equations III or IV. From equation III we obtain $M_0 = -0^{\text{m}}_{.22}$, and from equation IV $M_0 = -0^{\text{m}}_{.55}$. Since the value from the BOSS stars is the most probable, we thus get as the best values

$$(58) \quad M_0 = -0^{\text{m}}_{.2}, \quad \sigma = 0^{\text{m}}_{.9}.$$

D. OTHER ESTIMATES OF THE VALUES OF M_0 AND σ .

49. We will now try to estimate the values of M_0 and σ from other, independent sources. In the first place we may use the stars of the moving clusters of Ursa Major and Taurus.

Including in the Ursa Major system the stars given by HERTZSPRUNG¹⁾, the values of the absolute magnitudes, reduced to our scale, are given in table XVIII.

From these values of the absolute magnitudes we get

$$(59) \quad M_0 = -0.62, \quad \sigma = 0.97.$$

Excluding the most doubtful stars, β Aurigae and β Eridani, we get

$$(59^*) \quad M_0 = -0.88, \quad \sigma = 0.81.$$

TABLE XVIII.
THE URSA MAJOR SYSTEM.

Star	Spectrum	M
β Aurigae.....	A 0 p	- 2.60
ϵ Ursae maj. ...	A 0 p	- 1.75
α Coronae.....	A 0	- 1.19
β Eridani	A 2	- 1.01
ζ Ursae maj. A.	A 0 p	- 0.99
γ Ursae maj. ...	A 0	- 0.91
β Ursae maj. ...	A 0	- 0.77
α Canis maj. ...	A 0	- 0.21
δ Ursae maj. ...	A 0	+ 0.15
ζ Ursae maj. B.	A 2	+ 0.57
δ Leonis	A 2	+ 0.68
g Ursae maj. ...	A 5	+ 0.68

From the stars given by L. BOSS²⁾ as members of the Taurus stream, I have picked out those which are A-stars, in all 27 in number. These stars are given in table XIX.

TABLE XIX.
THE TAURUS STREAM.

BOSS N:r	Spectrum	m	M	BOSS N:r	Spectrum	m	M
935	A 0	5.48	+ 1.80	1043	A 5	5.97	+ 1.89
991	A 5	5.56	+ 0.70	1046	A 5	5.62	- 1.05
1004	A 0	5.59	+ 1.10	1047	A 5	5.12	+ 0.68
1007	A 0	5.27	+ 0.69	1051	A 0	5.70	+ 1.12
1013	A 0	5.76	+ 1.27	1054	A 5	4.84	+ 0.48
1018	A 2	5.68	+ 1.10	1056	A 0	5.49	+ 1.00
1022	A 2	4.84	+ 0.48	1067	A 5	4.75	+ 0.08
1026	A 3	4.86	- 0.40	1087	A 5	4.80	- 0.19
1027	A 2	5.42	+ 1.01	1090	A 0	5.15	+ 0.19
1029	A 0	4.24	- 0.48	1092	A 0	5.55	+ 0.59
1031	A 0	6.89	+ 1.81	1114	A 0	5.85	+ 0.94
1033	A 5	4.40	- 0.18	1143	A 5	5.12	+ 0.45
1034	A 0	4.60	+ 0.11	1226	A 0	5.42	+ 1.09
1040	A 0	5.74	+ 1.16				

The values of M are calculated from the parallaxes given by BOSS, with the help of the formula

$$(60) \quad M = m + 5 \log \pi + 3.48.$$

¹⁾ Ap. J. 30, 135, 1909.

²⁾ A. J. 26, 31, 1908.

The parallaxes in question lie between $0''.021$ and $0''.031$, with a mean of $0''.025$. A direct photographic determination of these parallaxes¹⁾ has given the mean value $0''.023 \pm 0.0025$ (p. e.), which affords a good check upon the reliability of the calculated parallaxes.

From the values of M given in the table we get

$$(61) \quad M_0 = +0.60, \quad \sigma = 0.65.$$

From 19 stars belonging to the group KAPTEYN²⁾ has got

$$(61^*) \quad M_0 = +0.69, \quad \sigma = 0.80.$$

As the stars belonging to the above star-streams are all apparent bright stars, it is very important to decide whether there are also any faint members or not. In the former case the values found above for M_0 will be too bright. For the Taurus stream the search for faint members has given a negative result, so that the value found for M_0 may be supposed to represent the real mean value for the stream. For the Ursa Major stream the question is still open, owing to the great extent of the group.

The values of M_0 and σ found from these two moving clusters are thus concordant with those found from the reduced proper motions and the mean parallax.

50. Concerning the parallax of the Taurus cluster KAPTEYN³⁾ remarks that in the above mentioned direct determination of the parallax, which was carried out at Groningen, the magnitude error offered considerable difficulty, and therefore it does not seem impossible that an error of $0''.01$, four times the probable error, may have crept into the result. Moreover, as the cluster is of a small extent, the determination of the convergent-point is rather uncertain. BOSS⁴⁾ has found as the value of the convergent-point

$$\alpha = 6^h 7^m 2, \quad \delta = +6^\circ 56',$$

but, according to KAPTEYN, the values

$$\alpha = 5^h 46^m, \quad \delta = +8^\circ 40'$$

satisfy the observations almost equally well. With this last value of the convergent KAPTEYN gets

$$\pi = 0''.033$$

and the corresponding mean value of the absolute magnitude is found to be

$$M_0 = +1.33 \text{ (reduced to our scale),}$$

a value more concordant with that found by KAPTEYN in another way.

¹⁾ G. P. 23.

²⁾ Ap. J. 47, 267, 1918.

³⁾ Loc. cit. page 267.

⁴⁾ Loc. cit.

51. In the paper referred to KAPTEYN has tried to determine the individual parallaxes of the B-stars, starting with the fact that the stars of this type show a certain disposition to form moving clusters. From these individual parallaxes the luminosity-curve is then derived. In this connection KAPTEYN has tried to determine the constants of the same curve for the A-stars in the following way. The bright B-stars show a particular crowding between $5^{\text{h}}20^{\text{m}}$ and $5^{\text{h}}40^{\text{m}}$ on both sides of the equator, and the same is the case with the faint B- and A-stars contained in the Draper Catalogue within the same limits. KAPTEYN now assumes that these stars form a local group, not extending much more in depth than laterally. As this group surrounds the famous Orion-nebula, it is called the nebula-group. For this group the proper motions are too small to give any reliable value for the parallax. But the frequency function of the apparent magnitudes for the group is different from that for the surrounding stars. The maximum of $a(m)$ seems to be displaced at the fainter side in the case of the nebula-group. The mean apparent magnitudes for the nebula-group and for the surrounding stars are estimated, and the difference found is wholly attributed to the difference in distance of the two groups. KAPTEYN thus gets the distance of the nebula-group in relation to the surrounding stars. From the brighter B-stars in the vicinity he has got $\pi = 0''.0081$ under the supposition that these B-stars form a moving cluster. The author now assumes that the faint A-stars have the same mean parallax as that found for the bright B-stars. (For the A-stars the corresponding mean parallax for $m = 8.43$, calculated from the formula given by KAPTEYN in G. P. 29, is $\pi = 0''.0032$ for $b = 0$.) Taking the mean value of m for the surrounding stars to be $8^{\text{m}}.43$ he gets

$$M_0 = +1^{\text{m}}.9, \quad \sigma = 0^{\text{m}}.9.$$

For the B 8—B 9 stars he obtains similarly

$$M_0 = +1^{\text{m}}.1, \quad \sigma = 0^{\text{m}}.8.$$

The dispersions thus found agree with that found by me, but the absolute magnitudes are about two magnitudes fainter. KAPTEYN's results however are, as we have seen, based on several assumptions. The reliability of these assumptions is open to doubt, and therefore we cannot attribute any very great weight to his values.

52. The direct parallax determinations for the A-stars are very few in number, on account of the relatively great distance of these stars. For the same reason the trustworthiness of the measured parallaxes is not great. In WALKEY's list of parallaxes in the Appendix to vol. XXVII of the *Journal of the British Astronomical Association* the measured parallaxes for 64 stars of types B 8 to A 5 are given. From these parallaxes I have calculated the absolute magnitudes with the help of the relation (60). Two variables are excluded, and the values of the absolute magnitudes of the remaining 62 stars are put together in table XX. To facilitate the computation of M_0 the stars are distributed into classes having a breadth of one magnitude. The mean of each class is given in column 1.

We get

$$M_0 = -0.39, \quad \sigma = 2.95.$$

Now the parallax-stars form a very selected material. The stars picked out for parallax measurement are partly those of great apparent brightness, partly those of

TABLE XX.

ABSOLUTE MAGNITUDES OF THE PARALLAX-STARS.

M	$F(M)$	$MF(M)$	$M^2 F(M)$	$(M+1)^2 F(M)$	Check
— 7.0	2	— 14	98	72	+ 548
— 6.0	1	— 6	36	25	— 48
— 5.0	2	— 10	50	32	+ 62
— 4.0	1	— 4	16	9	+ 562
— 3.0	6	— 18	54	24	
— 2.0	8	— 16	32	8	
— 1.0	11	— 11	11	0	
0.0	12	0	0	12	
+ 1.0	8	+ 8	8	32	
+ 2.0	3	+ 6	12	27	
+ 3.0	2	+ 6	18	32	
+ 4.0	1	+ 4	16	25	
+ 5.0	1	+ 5	25	36	
+ 6.0	3	+ 18	108	147	
+ 7.0	0	+ 0	0	0	
+ 8.0	1	+ 8	64	81	
Sum...	62	— 24	548	562	

great proper motion. In the former case the absolutely bright stars are favoured, in the second the absolutely faint ones. From the material in question, however, we get a mean value agreeing well with that found above. The dispersion comes out quite great. This is of course a consequence of the selection. The deviations from the mean are, however, greater than was to be expected from the dispersion found from the mean parallax.

53. For M_0 RUSSEL¹⁾ has obtained the following values:

Stars of measured parallax			Stars in cluster			Bright components of double stars	
Sp.	n	M_0	Sp.	n	M_0	Sp.	M_0
			B 8	8	— 1.3		
A 0	6	— 0.2	A 0	13	— 1.1	A 0	— 1.1
A 4	7	+ 0.9	A 4	26	+ 0.1	A 5	— 0.1.

These values agree with the M_0 found in this investigation.

54. A method of determining the individual parallaxes of the A-stars which differs from those mentioned before is given by PLUMMER²⁾. He assumes simply that these stars are moving in a direction parallel to the galactic plane. The following mean values of the absolute magnitudes are obtained.

¹⁾ Loc. cit. pp. 255, 283.

²⁾ Hypothetical Parallaxes of the brighter Stars of Type A. M. N. 72, 170, 1912. On the Motion and Distances of certain Stars of the Types B 8 and B 9. M. N. 72, 555, 1912.

Spectrum	n	M_0
B 8, B 9	30	— 2.2,
A	56	— 1.2,
A 1—A 3	37	— 0.3,
A 5—A 8	19	+ 1.0.

As the velocity surface for the A-stars has been found to be approximately an ellipsoid of revolution (compare e. g. the results in the first and fourth chapter of this investigation), the above assumption is only an approximation to the truth. Therefore about 22 % of the computed parallaxes turn out negative. As in the investigation only apparent bright stars are used, the above mean values are in all probability brighter than the real mean value and comparable with those given in table I.

55. In the course of his lectures during the past year CHARLIER has given another method for determining σ . From his study of the B-stars (L. M. II, 14) he found that for these stars the density function may be represented by

$$(62) \quad D(r) = D_0 e^{-\frac{r^2}{2\rho^2}}.$$

The same function is now assumed to be valid for stars of all types. For the frequency function of the absolute magnitude the form (9) is chosen. The writer has shown¹⁾ that under the above assumptions the function $a(m)$ may be given in the form of

$$(63) \quad a(m) = \psi(m) + 0.1768 \psi^{III}(m) + 0.0816 \psi^{IV}(m) + \dots$$

where

$$\psi(m) = \frac{N}{\alpha \sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\alpha^2}}.$$

If α is tolerably great we may approximately put

$$a(m) = \psi(m).$$

As has been shown by CHARLIER we have the following relations

$$(64) \quad m_0 = M_0 + 5 \cdot 10 \log \varrho + 0.7922,$$

$$(65) \quad \alpha^2 = \sigma^2 + 1.1020.$$

In § 48 we have got $\alpha = 2.2$, from formula (65) we should then get

$$\sigma = 1.79.$$

This value, however, is not reconcilable with the value found from the proper motions. With the value $\sigma = 1.79$ we get from equation III $M_0 = + 3.3$ and from equation IV $M_0 = + 1.4$.

¹⁾ Die Berechnung von Schiefeit und Exzeß in der Verteilung der scheinbaren Magnitude. (In the press.)

56. We have thus obtained from the reduced an common proper motions

$$M_0 = -0^m_2, \quad \sigma = 0^m_9.$$

The value found for M_0 agrees with that found from parallax- and cluster-stars. The latter also give a value of σ agreeing with that given above; the parallax-stars however suggest a greater dispersion. Now the value of σ is very difficult to determine. The value of this constant, as derived by me, is closely dependent on the proper motions and magnitudes of the fainter stars and is thus based on relatively poor material. Since moreover the value of M_0 is closely dependent on σ , the values found may be considered as only approximate. When the material becomes more abundant the method will give safer results.

The values found for M_0 and m_0 indicates a far greater extension towards the galactic poles than is found from the B-stars. With $M_0 = -0^m_2$ and $m_0 = 10^m_3$ we get—under the assumption that the form (52) for $a(m)$ is valid—half of the total number of stars outside

$$10^{0.2(10.3 + 0.2)} = 125 \text{ sirimeters,}$$

while for the B-stars CHARLIER has found a dispersion of 13 sirimeters in the direction of the galactic poles.

CHAPTER IV.

THE VELOCITY DISTRIBUTION AS DERIVED FROM THE REDUCED PROPER MOTIONS.

57. The problem of determining the distribution of the *linear* velocities from the proper motions was first solved by CHARLIER¹⁾. The solution was worked out under the assumption that the velocity-surface was an ellipsoid of revolution, and was afterwards generalized by WICKSELL²⁾ for the proper motions and by GYLLENBERG³⁾ for the radial velocities so as to embrace a three-axial ellipsoid. The generalized method was further simplified by CHARLIER, and this method was applied to the proper motions of the different spectral types by CHARLIER and GYLLENBERG⁴⁾. The methods thus developed also make it possible to pay regard to eventual deviations from the ellipsoidal distribution.

58. In the first cited memoir CHARLIER assumes the frequency function of the absolute magnitudes to have the form (9). For the density function he assumes the form derived in L. M. II, 8, *viz.*

$$(66) \quad dr r^2 D(r) = - \frac{dy}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(y-y_0)^2}{2\sigma_1^2}}. \quad \left(y = -\frac{1}{b} \log r. \right)$$

Under these assumptions he got, with the help of (7), the mean value of $\frac{1}{r^s}$ for a given m in the form of

$$\mathfrak{A}_s = \left(\frac{1}{r^s} \right)_m = p^s \cdot q^{s^2},$$

or, introducing the mean parallax $\mathfrak{A}_1 = p \cdot q$

$$\mathfrak{A}_s = \mathfrak{A}_1^s q^{s^2 - s}.$$

Introducing $q' = \frac{1}{q^s}$ we get

$$(67) \quad \mathfrak{A}_s = \mathfrak{A}_1^s q'^{\frac{1}{2}(s-s^2)}.$$

With the help of this relation it has been shown by CHARLIER that the moments of any order of the linear motions may be obtained from the proper motions as soon

¹⁾ *Studies in Stellar Statistics* II. The Motion of the Stars. L. M. II, 9, 1913.

²⁾ L. M. II, 12.

³⁾ L. M. II, 13.

⁴⁾ L. M. I, 92.

as the values of the constants $\mathfrak{P}_1$ and q' have been obtained. $\mathfrak{P}_1$, the mean parallax, is obtained from the moments of the first order, but the determination of the constant q' , which is dependent on the forms of the density and luminosity functions of the stars in question, offers considerable difficulty. According to CHARLIER we have

$$q' = e^{-b^2 \alpha^2 \lambda_1 (1 - \lambda_1)}$$

where α is the dispersion in the apparent magnitudes and λ_1 a constant defined in § 25.

Now it has been shown by WICKSELL (loc. cit.) that for a value of q' about 0.75 the characteristics of the third and fourth orders vanish. As the values of these characteristics from the radial velocities are found to be small (GYLLENBERG, loc. cit.), it seems probable that the value of q' —when all types are taken together—lies about 0.75. In L. M. I, 92 CHARLIER and GYLLENBERG have tried to determine the value of q' for each spectral type separately by a comparison of the squares of the total mean velocity obtained from the proper motions and from the radial velocities. The results thus obtained indicate that q' is probably independent of the spectral type. As a mean value they get $q' = 0.675$.

59. The problem of determining the velocity distribution from the reduced proper motions may be treated in a quite analogous manner. For the F-stars the computation has already been performed by LUNDAHL in L. M. II, 21. Under the same assumptions as above regarding the density and luminosity functions the moments of the linear velocities may be calculated from the reduced proper motions with the help of the mean reduced parallax and a constant q' , which was shown by LUNDAHL (loc. cit.) to be identical with the corresponding quantity for the common proper motions. But the proof given by LUNDAHL is only valid under the assumption of constant m , and in this case the proof may be generalized so as to be valid independent of the forms of the density and luminosity functions. Then according to (1*) we have

$$r = R \cdot e^{b m}$$

from which follows

$$(68) \quad \frac{1}{r^s} = \frac{1}{R^s} \cdot e^{-s b m}.$$

Putting, for the sake of symmetry, $\left(\frac{1}{R^s}\right) = \Theta_s$ it follows from (68) that, when m is constant,

$$(69) \quad \mathfrak{P}_s = \Theta_s \cdot e^{-s b m}$$

and

$$\mathfrak{P}_s : \mathfrak{P}_1^s = \Theta_s : \Theta_1^s. \quad \text{H. E. D.}$$

60. Now the relation (67) was deduced by CHARLIER under the assumption of a constant m , but the method was applied to stars brighter than a given m . It has been shown by WICKSELL (loc. cit.) that in this case we have for the common proper motions approximately an analogous relation at least as long as only the brighter stars are considered.

As for stars brighter than a given m the values of q' for the common and reduced proper motions cannot be considered as equal.

Using the form (9) of $\varphi(M)$ and (66) of $D(r)$ we may, for stars brighter than m , easily deduce the relation

$$(70) \quad \mathcal{P}_s = e^{\frac{s b y_0 + \frac{1}{2} s^2 b^2 \sigma_1^2}{A(m)}}$$

and from (16) we have

$$(71) \quad \Theta_s = e^{\frac{s b M_0 + \frac{1}{2} s^2 b^2 \sigma^2}{A(m)}}$$

Considering now the special case if we had treated the stars of all apparent magnitudes. From the common proper motions we should thus have

$$\begin{aligned} \mathcal{P}_s &= e^{\frac{s b y_0 + \frac{1}{2} s^2 b^2 \sigma_1^2}{A(m)}} \\ &= \mathcal{P}_1^s e^{\frac{1}{2} b^2 \sigma_1^2 (s^2 - s)}, \quad \therefore q' = e^{-b^2 \sigma_1^2} \end{aligned}$$

and from the reduced motions

$$\begin{aligned} \Theta_s &= e^{\frac{s b M_0 + \frac{1}{2} s^2 b^2 \sigma^2}{A(m)}} \\ &= \Theta_1^s e^{\frac{1}{2} b^2 \sigma^2 (s^2 - s)}, \quad \therefore q' = e^{-b^2 \sigma^2}. \end{aligned}$$

The two values of q' are not equal, the former being dependent only on the *density* function, the latter only on the *luminosity* function.

61. When determining the velocity ellipsoid from the reduced proper motions, the method given in L. M. I, 92 is followed. We start from the equations (34). Squaring these equations, we get

$$\begin{aligned} (72) \quad u_0'^2 &= \beta_{11}^2 \left(\frac{U_g}{R} \right)^2 + \beta_{21}^2 \left(\frac{V_g}{R} \right)^2 + 2 \beta_{11} \beta_{21} \frac{U_g}{R} \cdot \frac{V_g}{R}, \\ v_0'^2 &= \beta_{12}^2 \left(\frac{U_g}{R} \right)^2 + \beta_{22}^2 \left(\frac{V_g}{R} \right)^2 + \beta_{32}^2 \left(\frac{W_g}{R} \right)^2 + 2 \beta_{22} \beta_{32} \frac{V_g}{R} \cdot \frac{W_g}{R} + \\ &\quad + 2 \beta_{32} \beta_{12} \frac{W_g}{R} \cdot \frac{U_g}{R} + 2 \beta_{12} \beta_{22} \frac{U_g}{R} \cdot \frac{V_g}{R}. \end{aligned}$$

Multiplying the two equations (34) we have, too,

$$\begin{aligned} (72^*) \quad u_0' v_0' &= \beta_{11} \beta_{12} \left(\frac{U_g}{R} \right)^2 + \beta_{21} \beta_{22} \left(\frac{V_g}{R} \right)^2 + \beta_{21} \beta_{32} \frac{V_g}{R} \cdot \frac{W_g}{R} + \beta_{32} \beta_{11} \frac{W_g}{R} \cdot \frac{U_g}{R} + \\ &\quad + (\beta_{11} \beta_{22} + \beta_{12} \beta_{21}) \frac{U_g}{R} \cdot \frac{V_g}{R}. \end{aligned}$$

Taking the means for every square and introducing the moments about the origin through the relations

$$\begin{aligned}\nu'_{20} &= (\overline{u_0'^2}), & \nu'_{02} &= (\overline{v_0'^2}), & \nu'_{11} &= (\overline{u_0' v_0'}), \\ N'_{200} &= (\overline{U_g^3}), & N'_{020} &= (\overline{V_g^3}), & N'_{002} &= (\overline{W_g^3}), \\ N'_{011} &= (\overline{V_g W_g}), & N'_{101} &= (\overline{W_g U_g}), & N'_{110} &= (\overline{U_g V_g}),\end{aligned}$$

we get, putting $\left(\frac{1}{R}\right) = \Theta$, and $\left(\frac{1}{R^2}\right) = \Theta_1^2 : q'$ and observing that we suppose the linear velocities to be independent of R ,

$$\begin{aligned}\nu'_{20} q' &= \beta_{11}^2 \Theta_1^2 N'_{200} + \beta_{21} \Theta_1^2 N'_{020} + 2 \beta_{11} \beta_{21} \Theta_1^2 N'_{110}, \\ \nu'_{02} q' &= \beta_{12}^2 \Theta_1^2 N'_{200} + \beta_{22}^2 \Theta_1^2 N'_{020} + \beta_{32}^2 \Theta_1^2 N'_{002} + 2 \beta_{22} \beta_{32} \Theta_1^2 N'_{011} + \\ (73) \quad &+ 2 \beta_{32} \beta_{12} \Theta_1^2 N'_{101} + 2 \beta_{12} \beta_{22} \Theta_1^2 N'_{110}, \\ \nu'_{11} q' &= \beta_{11} \beta_{12} \Theta_1^2 N'_{200} + \beta_{21} \beta_{22} \Theta_1^2 N'_{020} + \beta_{21} \beta_{32} \Theta_1^2 N'_{011} + \beta_{32} \beta_{11} \Theta_1^2 N'_{101} + \\ &+ (\beta_{11} \beta_{22} + \beta_{12} \beta_{21}) \Theta_1^2 N'_{110}.\end{aligned}$$

But the moments about the mean of the linear motions are obtained from the relations

$$\begin{aligned}N_{200}^G &= N'_{200} - \overline{U_g^2}, & N_{020}^G &= N'_{020} - \overline{V_g^2}, & N_{002}^G &= N'_{002} - \overline{W_g^2}, \\ N_{011}^G &= N'_{011} - \overline{V_g \cdot W_g}, & N_{101}^G &= N'_{101} - \overline{W_g \cdot U_g}, & N_{110}^G &= N'_{110} - \overline{U_g \cdot V_g}.\end{aligned}$$

With the help of these relations we get from the equations (73), making use of (34*), the following three equations for each square:—

$$\begin{aligned}\nu'_{20} q' - x_0'^2 &= \beta_{11}^2 \Theta_1^2 N_{200}^G + \beta_{21} \Theta_1^2 N_{020}^G + 2 \beta_{11} \beta_{21} \Theta_1^2 N_{110}^G, \\ \nu'_{02} q' - y_0'^2 &= \beta_{12}^2 \Theta_1^2 N_{200}^G + \beta_{22}^2 \Theta_1^2 N_{020}^G + \beta_{32}^2 \Theta_1^2 N_{002}^G + 2 \beta_{22} \beta_{32} \Theta_1^2 N_{011}^G + \\ (74) \quad &+ 2 \beta_{32} \beta_{12} \Theta_1^2 N_{101}^G + 2 \beta_{12} \beta_{22} \Theta_1^2 N_{110}^G, \\ \nu'_{11} q' - x_0' y_0' &= \beta_{11} \beta_{12} \Theta_1^2 N_{200}^G + \beta_{21} \beta_{22} \Theta_1^2 N_{020}^G + \beta_{21} \beta_{32} \Theta_1^2 N_{011}^G + \\ &+ \beta_{32} \beta_{11} \Theta_1^2 N_{101}^G + (\beta_{11} \beta_{22} + \beta_{12} \beta_{21}) \Theta_1^2 N_{110}^G,\end{aligned}$$

where $x_0' = \overline{u_0'}$, $y_0' = \overline{v_0'}$.

Treating these systems of equations with the method of least squares we get the moments about the mean multiplied by Θ_1^2 as linear functions of the parameter q' . Now Θ_1 was determined in the third chapter, and thus we get the values of the moments about the mean of the linear motions referred to our galactic system of coordinates as soon as the value of the parameter q' has been settled. When the moments have been determined the magnitudes and directions of the axes of the velocity ellipsoid may be determined with the help of the method given by CHARLIER in L. M. I, 66 and used by me in L. M. I, 76. However, by a method given by WICKSELL¹⁾ the computations may be considerably shortened. I have therefore used his method here.

¹⁾ Beiträge zum Studium der BRAVAIS'schen Function. (Forthcoming in L. M.)

From the moments we form the *Jacobian* determinant

$$J(s) = \begin{vmatrix} N_{200}^G - s, & N_{110}^G, & N_{101}^G \\ N_{110}^G, & N_{020}^G - s, & N_{011}^G \\ N_{101}^G, & N_{011}^G, & N_{002}^G - s \end{vmatrix} = 0.$$

This determinant defines an equation in s of the third degree. The three roots of this equation at once give us the squares of the principal axes. The direction cosines l_i, m_i, n_i of the axis corresponding to the root s_i are then obtained from the equations

$$\begin{aligned} (N_{200}^G - s_i) l_i + N_{110}^G m_i + N_{101}^G n_i &= 0, \\ N_{110}^G l_i + (N_{020}^G - s_i) m_i + N_{011}^G n_i &= 0, \\ N_{101}^G l_i + N_{011}^G m_i + (N_{002}^G - s_i) n_i &= 0. \end{aligned}$$

62. Using this method the velocity ellipsoid of the stars of the subclasses B 8 to A 5 brighter than the sixth magnitude is computed from the reduced proper motions. The proper motions having been taken from the catalogue of BOSS, the material is the same as that used in section A of the third chapter.

When forming the moments of the reduced proper motions for each square we again meet with the difficulty arising from the presence of large motions in the material. Owing to the facts mentioned in § 30 I have considered it correct not to exclude any star on account of large reduced proper motion when calculating the mean reduced parallax. But in the case of the moments of the second order the matter is somewhat different. As long as only the moments of the first order were considered the influence of the large motions is of an accidental character, but when forming the second order moments we have to square each individual motions, and thus all large motions will act in the same direction, *viz.* so as to enlarge the values of the moments. In this case we have to deal with a *systematical* influence.

To study the influence of the large motions upon the velocity distribution I have therefore performed the calculations for two different cases, first without making any exclusion of large motions (Material A), secondly after exclusion according to the principles given in § 32 (Material B).

The moments about the origin of the reduced proper motions for each double-square are given in tables XXI (Material A) and XXII (Material B).

We get from material A

$$\begin{aligned} \Theta_1^2 N_{200}^G &= 17.502 q' - 5.310, & \Theta_1^2 N_{110}^G &= -0.556 q' - 1.495, \\ \Theta_1^2 N_{020}^G &= 4.514 q' - 0.188, & \Theta_1^2 N_{011}^G &= +1.085 q' - 0.501, \\ \Theta_1^2 N_{002}^G &= 3.067 q' - 0.754, & \Theta_1^2 N_{101}^G &= +2.054 q' - 2.042, \end{aligned}$$

and from material B

$$\begin{aligned} \Theta_1^2 N_{200}^G &= 13.743 q' - 5.223, & \Theta_1^2 N_{110}^G &= +0.878 - 1.890, \\ \Theta_1^2 N_{020}^G &= 2.553 q' - 0.123, & \Theta_1^2 N_{011}^G &= +0.673 - 0.440, \\ \Theta_1^2 N_{002}^G &= 1.987 q' - 0.543, & \Theta_1^2 N_{101}^G &= +1.637 - 1.608. \end{aligned}$$

The unknowns are expressed in radians per stellar year.

TABLE XXI.
THE MOMENTS ABOUT THE ORIGIN. MATERIAL A.

Square	N	x_0	y_0	ν'_{20}	ν'_{02}	ν'_{11}	Square	N	x_0	y_0	ν'_{20}	ν'_{02}	ν'_{11}
GA_1	34	+0.721	+0.046	+1.1841	+0.1551	-0.0612	GC_1	86	-0.086	-0.067	+0.2220	+0.0681	+0.0241
GA_2	38	-0.320	-0.133	+0.5747	+0.1869	+0.0211	GC_2	102	+0.144	-0.135	+0.3654	+0.0724	-0.0357
GB_1	45	-0.003	+0.140	+0.4054	+0.4144	-0.0055	GC_3	89	+0.345	-0.168	+0.3890	+0.0946	-0.0878
GB_2	46	+0.290	+0.111	+0.6881	+0.2208	+0.1837	GC_4	75	+0.485	-0.171	+0.7801	+0.1089	-0.1809
GB_3	43	+0.431	-0.026	+0.5970	+0.1166	-0.0324	GC_5	67	+0.377	-0.227	+0.3547	+0.1947	-0.1846
GB_4	33	+0.402	-0.242	+0.4596	+0.2981	-0.2509	GC_6	59	+0.243	-0.325	+0.1969	+0.3291	-0.1885
GB_5	56	+0.303	-0.404	+0.2914	+0.4211	-0.1921	GC_7	70	-0.040	-0.218	+0.1788	+0.1200	+0.0147
GB_6	50	-0.035	-0.385	+0.4431	+0.3977	+0.0793	GC_8	50	-0.246	-0.325	+0.4449	+0.2344	+0.2227
GB_7	41	-0.227	-0.402	+0.5099	+0.4174	+0.2246	GC_9	76	-0.314	-0.304	+0.7350	+0.2077	-0.0238
GB_8	35	-0.623	-0.280	+1.2577	+0.4195	-0.1929	GC_{10}	69	-0.498	-0.167	+0.5634	+0.0990	+0.0482
GB_9	43	-0.509	-0.028	+0.5872	+0.1987	-0.1150	GC_{11}	79	-0.409	-0.172	+0.3181	+0.1479	+0.0422
GB_{10}	60	-0.204	+0.014	+0.1285	+0.2171	-0.0464	GC_{12}	88	-0.248	-0.021	+0.1526	+0.0867	0.0000

TABLE XXII.
THE MOMENTS ABOUT THE ORIGIN. MATERIAL B.

Square	N	x_0	y_0	ν'_{20}	ν'_{02}	ν'_{11}	Square	N	x_0	y_0	ν'_{20}	ν'_{02}	ν'_{11}
GA_1	33	+0.649	+0.086	+0.8996	+0.1109	+0.0603	GC_1	85	-0.057	-0.065	+0.1481	+0.0682	+0.0169
GA_2	38	-0.320	-0.133	+0.5747	+0.1869	+0.0211	GC_2	101	+0.099	-0.129	+0.1580	+0.0684	-0.0068
GB_1	43	-0.036	+0.224	+0.1970	+0.2882	-0.0095	GC_3	87	+0.321	-0.143	+0.2561	+0.0611	-0.0449
GB_2	43	+0.234	+0.163	+0.4417	+0.1702	+0.1612	GC_4	70	+0.490	-0.132	+0.5040	+0.0714	-0.0967
GB_3	43	+0.431	-0.026	+0.5970	+0.1166	-0.0324	GC_5	63	+0.293	-0.156	+0.1937	+0.0774	-0.0530
GB_4	32	+0.379	-0.195	+0.4319	+0.2117	-0.1953	GC_6	57	+0.202	-0.258	+0.1330	+0.1370	-0.0778
GB_5	55	+0.270	-0.405	+0.2142	+0.4263	-0.1813	GC_7	67	-0.051	-0.219	+0.1153	+0.0949	-0.0213
GB_6	47	-0.066	-0.344	+0.1434	+0.3393	-0.0760	GC_8	47	-0.165	-0.258	+0.3033	+0.1277	+0.1059
GB_7	39	-0.133	-0.341	+0.2932	+0.2304	+0.1154	GC_9	73	-0.396	-0.270	+0.5031	+0.1215	+0.1389
GB_8	34	-0.738	-0.196	+0.9878	+0.1589	+0.0899	GC_{10}	67	-0.488	-0.141	+0.4901	+0.1263	+0.1229
GB_9	42	-0.475	-0.022	+0.5113	+0.2006	-0.1296	GC_{11}	75	-0.404	-0.147	+0.2762	+0.0685	+0.0627
GB_{10}	60	-0.204	+0.014	+0.1285	+0.2171	-0.0464	GC_{12}	85	-0.221	-0.039	+0.1129	+0.0504	-0.0012

63. The values of Θ_1 , the mean reduced parallax, were determined in the third chapter. In § 31 we got from material A the value

$$\Theta_1 = \left(\frac{1}{R} \right) = 0.584 \pm 0.051$$

and in § 32 we got from material B

$$\Theta_1 = \left(\frac{1}{R} \right) = 0.553 \pm 0.050.$$

To get the linear moments we have to fix the value of the constant q' .

When treating the reduced proper motions of the F-stars LUNDAHL (loc. cit.) has used a method for determining q' similar to that used by CHARLIER and GYLLENBERG, viz. a comparison of the second order moments obtained from the reduced proper motions with those obtained from the radial velocities. If we use the same method here we get from material A $q' = 0.63$ and from material B $q' = 0.79$. Now these values are

only approximate, as we have used the reduced proper motions for the B 8—A 5 stars and compared the total mean velocity thus obtained with that obtained by GYLLENBERG¹⁾ from the radial velocities of the A 0—A 5 stars. One fact comes out of this comparison—the value of q' , determined in this way, is very dependent on the mode of excluding large motions.

However, with the help of the formulae deduced in the second chapter, we may calculate the value of Θ_s as soon as the mean value and the dispersion of the absolute magnitude are known. We get, indeed, from (17)

$$(75) \quad \Theta_s = \left(\frac{1}{R^s} \right) = e^{s b M_0 + \frac{1}{2} s^2 b^2 \sigma^2} \frac{A(m - s b \sigma_2)}{A(m)}.$$

With the form of $A(m)$ found for all stars brighter than the sixth magnitude in § 33 we get from (75)

$$(76) \quad q' = e^{-b^2 \sigma^2 - 0.30 b^3 \sigma^4}$$

with the value $\sigma = 0.86$ found in § 48 we get from (76)

$$q' = 0.84.$$

64. According to the uncertainty of the value of q' I have performed the computations of the axes of the velocity ellipsoid and their directions for three different values of q' , viz. $q' = 1.00$, 0.85 and 0.75 . The results obtained are put together in table XXIII for material A, and table XXIV for material B.

TABLE XXIII.

THE LENGTHS OF THE AXES OF THE VELOCITY ELLIPSOID AND THEIR DIRECTIONS. MATERIAL A.

q'	Vertex I			Vertex II			Vertex III		
	a_1	l	b	a_2	l	b	a_3	l	b
1.00	6.10	346°2	—0°6	3.42	76°0	+17°3	2.51	258°2	+72°7
0.85	5.46	343°1	—2°7	3.02	72°5	+12°3	2.28	265°4	+77°5
0.75	5.01	340°1	—4°6	2.72	69°6	+ 6°7	2.09	292°9	+83°2

TABLE XXIV.

THE LENGTHS OF THE AXES OF THE VELOCITY ELLIPSOID AND THEIR DIRECTIONS. MATERIAL B.

q'	Vertex I			Vertex II			Vertex III		
	a_1	l	b	a_2	l	b	a_3	l	b
1.00	5.29	355°2	+0°1	2.83	85°2	+12°6	2.09	294°9	+78°2
0.85	4.63	351°8	—2°4	2.56	81°5	+ 6°2	1.85	282°8	+83°3
0.75	4.15	348°0	—5°1	2.31	79°1	— 1°2	1.69	1°2	+84°8

¹⁾ L. M. II, 13, page 42.

If we calculate the ratio of the three principal axes, we get the following values.

q'	Material A	Material B
1.00	2.48 : 1.36 : 1.00	2.53 : 1.35 : 1.00,
0.85	2.39 : 1.32 : 1.00	2.50 : 1.38 : 1.00,
0.75	2.40 : 1.30 : 1.00	2.46 : 1.37 : 1.00.

The form of the velocity ellipsoid thus seems to be practically independent of the value of q' , in conformity with the results obtained by WICKSELL. By a comparison of the results from material A and B we see that the exclusion of large motions does not materially alter the form of the ellipsoid. It is chiefly the *lengths* of the principal axes which are affected by a variation of q' and an exclusion of the large motions.

Observatory Lund, mars 1920.

When finishing I wish to express my heartfelt thanks to Professor C. V. L. CHARLIER, who has given rise to this investigation and who has followed my work with his warmest interest and done everything to facilitate it. To the Astronomer Royal, Sir F. W. DYSON I also wish to express my sincere thanks for his kindness to place at my disposal the still unpublished material of the *Greenwich Catalogue*. My colleagues, former and present, I also take the opportunity of thanking for many valuable suggestions, given to me in the course of my investigation.

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